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ELEMENTS

OF

ARITHMETIC.

LONDON:
PRINTED BY THOMAS DAVISON, WHITEFRIARS.

S.4-1830c

THE

ELEMENTS

OF

ARITHMETIC.

BY

AUGUSTUS DE MORGAN,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF LONDON.

> "Ce n'est point par la routine qu'on s'instruit, c'est par sa propre réflexion; et il est essentiel de contracter l'habitude de se rendre raison de ce qu'on fait: cette habitude s'acquiert plus facilement qu'on ne pense; et une fois acquise, elle ne se perd plus."
>
> CONDILLAC.

LONDON:
PRINTED FOR JOHN TAYLOR,
BOOKSELLER AND PUBLISHER TO THE UNIVERSITY OF LONDON,
30, UPPER GOWER-STREET.
1830

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BIBLIOTHECA·BODLEIANA·
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ILLU MEA

PREFACE.

THIS little work is an attempt to give the young student the common rules of Arithmetic, accompanied by the reasoning to which he must habituate his mind before he can make progress in any science.

I might speak from experience, of the nature of the arithmetical knowledge which most youths acquire before they commence the study of Geometry and Algebra. But as almost all agree in opinion, that this science ought not to be, as it is in this country, degraded into a mass of rules learned by rote, one half of which are of no use but in commercial business, and rarely even there, I will proceed to make a remark on the manner in which this book should be studied.

In order to avoid the generalities of algebraic language, which the mind of a beginner cannot grasp, it is necessary to confine each demonstration to one particular case; that is, to show, on some particular numbers, those truths which, in Algebra, are asserted of all at once, by means of letters employed to stand for numbers. From the case which is chosen, a rule is drawn which is assumed to hold good always. This reasoning is not strictly logical; but it must be recollected, that the student has it in his power to convince himself of the universal truth of what is stated, by employing different numbers from those used in the text, in every demonstration. This is what I recommend him to do: if he omits this exercise, he does not give the subject a fair trial.

With respect to Commercial Arithmetic, I have introduced every rule and principle which is essential in the routine of the counting-house. At the same time, nothing is really omitted which is not a simple deduction from the principles laid down in the second book of this treatise. To instruct the future man of business in all the minutiæ of mercantile computations was not my object in writing this work. That object has been to reason the student into a knowledge of those rules which are usually received from the authority of a book, and to make the study of Arithmetic a development of the powers of the understanding.

There are some who defend the present system of teaching this science upon the ground that children retain and apply a rule with more facility than a principle or an argument. They alter Newton's celebrated maxim, and make it stand thus: "In scientiis ediscendis magis prosunt exempla *sine præceptis.*" This I am not disposed to grant; nor do I think that its supporters can themselves derive much satisfaction from observing the effects of their method, or can assert that Arithmetic is well taught to the majority of educated persons, even in the seven or eight years during which they are employed in toiling from rule to rule through countless examples, and dragging at each remove a lengthened chain. But let their argument be allowed, even to the extent of supposing that a child can retain ten rules with more ease than one principle, it still behoves them to consider well the import of the answer which the Lioness made to the Fox, as recorded by Æsop.

To the parents and preceptors of youths intended for the University of London I can recommend this book without presumption; because it is admitted that one system of instruction pursued throughout, even though that should not be the best, is preferable to the change of methods and habits which the student usually suffers in passing from one system to another. I can assure them, that a thorough knowledge of the principles and practice contained in the following pages will constitute a better preparation for the business of my lecture-room than that which is usually made, viz., the study of some thirty or forty propositions of Euclid, and a few algebraic rules, not reasonings, built upon no knowledge whatever of the principles of Arithmetic, or of its practice beyond the first four rules.

AUGUSTUS DE MORGAN.

UNIVERSITY OF LONDON,
March 19*th*, 1830.

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BOOK I.

PRINCIPLES OF ARITHMETIC.

SECTION I.

NUMERATION.

(1.) IMAGINE a multitude of objects of the same kind assembled together, for example, a company of horsemen. One of the first things that must strike a spectator, although unused to counting, is, that to each man there is a horse. Now, though men and horses are things perfectly unlike, yet a connexion between them will be formed in the mind of the observer, which he will show by saying that there is the same *number* of men as of horses. Again, suppose that there are two companies of horsemen, and he wishes to know in which of them is the greater number, and also to be able to recollect how many there are in each—

(2.) Suppose that, while the first company passes by, he drops a pebble into a basket for each man that passes him. There is no connexion between the pebbles and the horsemen but this, that for every horseman there is a pebble, that is, in common language, the *number* of pebbles and of horsemen is the same. Suppose that while the second company passes he drops a pebble for each man into a second basket; he will then have two baskets of pebbles, by which he will be able to make any other person know how many horsemen there were in each company. Suppose he wishes to know which company was the larger, or contained most horsemen; he will take a pebble out of each basket, and put them aside. He will go on doing this as often as he can, that is, until one of the baskets is emptied. Then for each pebble which is left in the other basket there is a horseman more in the larger company than there is in the smaller.

(3.) In this way a savage could reckon every thing which is necessary for him. He could thus register his children, his cattle, the number of summers and winters which he had seen, by means of pebbles, or any other things which could be got in large numbers. Something of this sort is the practice of savage nations at this day, and it has in some places lasted even after better methods of reckoning had been invented. At Rome, in the time of the republic, the prætor, one of the magistrates, used to go every year in great pomp, and drive a nail into the door of the temple of Jupiter.

(4.) In process of time the savage would give names to those collections of pebbles which he met with most frequently, that is, to all small numbers of pebbles. But to him the most convenient way of reckoning small numbers would be on his fingers. He could make upon his fingers the little calculations which would be necessary for his purposes, and would name all the different collections of the fingers. He would thus get names in his own language answering to one, two, three, four, five, six, seven, eight, nine, and ten. As his wants increased he would find it necessary to give names to larger numbers; but here he would be stopped by the immense quantity of names which he must have in order to express all the numbers which he would be obliged to make use of. He must then, after giving a separate name to a few of the first numbers, manage to express all other numbers by means of those names.

(5.) I now show how this has been done in our own language. The English names of numbers have been formed from the Saxon; and if each number be written down in one column, and its meaning in another, the way in which our numbers are expressed will be evident.

One
two
three
four
five
six
seven

eight	
nine	
ten	
eleven	one left [*]
twelve	two left
thirteen	ten and three
fourteen	ten and four
fifteen	ten and five
sixteen	ten and six
seventeen	ten and seven
eighteen	ten and eight
nineteen	ten and nine
twenty	two tens
twenty-one	two tens and one
twenty-two	two tens and two
&c. &c.	&c. &c.
thirty	three tens
&c.	&c.
forty	four tens
&c.	&c.
fifty	five tens
sixty	six tens
seventy	seven tens
eighty	eight tens
ninety	nine tens
a hundred	ten tens
a hundred and one	ten tens and one
&c. &c.	
a thousand	ten hundreds

---

[*] Meaning one left after ten is taken, or ten and one.

ten thousand
a hundred thousand
a million ten hundred thousand
or one thousand thousand
ten millions
a hundred millions
&c.

(6.) Words, such as have been just written down, would very soon be too long for the continual repetition which would take place in writing calculations. Short signs would then be substituted for words; but it would be impossible to have a separate sign for every separate number, so that when a small number of signs had been invented, it would be convenient to invent others for the rest out of those already made. The signs which we use are as follows:

0	nothing
1	one
2	two
3	three
4	four
5	five
6	six
7	seven
8	eight
9	nine

I now proceed to explain how these signs are made to represent other numbers.

(7.) Suppose a man to hold up, first one finger, then two, and so on, until he has held up every finger, and suppose any number of other men to do the same thing. It is plain that we may thus distinguish one number from another, by causing two different sets of persons to hold up each a certain number of fingers, and that we may do this in a great many different ways. For example, the number fifteen would be indicated either by fifteen men each holding up one finger, or by four men each holding up two fingers, and a fifth holding up seven, and so on in other different ways. Now the question is, of all these ways of expressing the number, which is the most convenient? In the choice which is made for this purpose consists what is called the method of numeration.

(8.) I have used the foregoing explanation because it is very probable that our system of numeration, and almost every other which is used in the world, began from the practice of reckoning on the fingers, which children usually follow when first they begin to count. The method which I have described is the rudest possible; but, by a little alteration, a system may be formed which will enable us to express enormous numbers with great ease.

(9.) Suppose that you are going to count some large number, for example, to measure a number of yards of cloth. Opposite to yourself, on your right hand, suppose a man to be placed who keeps his eye upon you, and when he sees that you have measured the first yard holds up one finger, and holds up another finger for each other yard which he sees you measure. When ten yards have been measured he will have held up ten fingers, and will not be able to count any further unless he begin again holding up one finger at the eleventh yard, two at the twelfth, and so on. But to know how many have been counted, you must know not only how many fingers he holds up, but also how many times he has begun again. You may know this by placing another man at his right hand, who keeps his eye, not upon you, but upon his companion, and holds up one finger the moment he perceives him ready to begin again, that is, as soon as ten yards have been measured. You see that each finger of the first man stands only for one yard, but that each finger of the second stands for as many as all the fingers of the first together, that is, for ten. In this way a hundred may be counted, because the first may now reckon his ten fingers once for each finger of the second man, that is ten times in all, and ten tens is one hundred (5). Now place a third man at the right of the second, who shall hold up a finger whenever he perceives the second ready to begin again. You see that one finger of the third man counts as many as all the ten fingers of the second, that is, counts one hundred. In this way we may proceed until the third has all his fingers extended, which will signify that ten hundred or one thousand have been counted (5). A fourth man would enable us to count as far as ten thousand, a fifth as far as one hundred thousand, a sixth as far as a million, and so on.

(10.) Since each new person was supposed to stand to the right of the one who came before him, and since they all stand opposite to you, the new ones have taken their places upon your left. Now rule a number of columns as is done in the next page, and to the right of them all place in words the number which you wish to represent; in the first column to the right place the number of fingers which the first man will be holding up when that number of yards has been measured. In the next column place the fingers which the second man will then be holding up, and so on.

	7th.	6th.	5th.	4th.	3d.	2d.	1st.	
I.						5	7	fifty-seven.
II.					1	0	4	one hundred and four.
III.					1	1	0	one hundred and ten.
IV.				2	3	4	8	two thousand three hundred and forty-eight.
V.			1	5	9	0	6	fifteen thousand nine hundred and six.
VI.		1	8	7	0	0	4	one hundred and eighty-seven thousand & four.
VII.	3	6	9	7	2	8	5	three million, six hundred and ninety-seven thousand, two hundred and eighty-five.

(11.) In I. the number fifty-seven is expressed. This means (from 5) five tens and seven. The first has therefore counted all his fingers five times, and has counted seven fingers more. This is shown by five fingers of the second man being held up, and seven of the first. In II. the number one hundred and four is represented. This number is (5) ten tens and four. The second person has therefore just reckoned all his fingers once, which is denoted by the third person holding up one finger; but he has not yet begun again, because he does not hold up a finger until the first has counted ten, of which ten only four have been counted. When all this last-mentioned ten have been counted, he then holds up one finger, and the first being ready to begin again, has no fingers extended. This is the case in III. You will now find no difficulty with the other numbers in the table.

(12.) In all these numbers a figure in the first column only stands for as many yards as are written under that figure in (6). A figure in the second column stands, not for as many yards, but for as many tens of yards: a figure in the third column stands for as many hundreds of yards; in the fourth column for as many thousands of yards; and so on: that is, if we suppose a figure to move from any column to the one on its left, it stands for ten times as many yards as before. Recollect this, and you may cease to draw the lines between the columns, because each figure will be sufficiently well known by the *place* in which it is; that is, by the number of figures which come upon the right hand of it.

(13.) The things here measured were yards of cloth. In this case one yard of cloth is called the *unit*. The first figure on the right is said to be in the *units' place*, because it only stands for so many units as are in the number that is written under it in (6). The second figure is said to be in the *tens'* place, because it stands for a number of tens of units. The third, fourth, and fifth figures are in the places of the *hundreds*, *thousands*, and *tens of thousands*, for a similar reason.

(14.) If the quantity measured had been acres of land, an acre of land would have been called the *unit*, for the unit is *one* of the things which are measured. Quantities are of two sorts; those which contain an exact number of units, as 47 yards, and those which do not, as 47 yards and a half. Of these, for the present we only consider the first.

(15.) In most parts of arithmetic all quantities must have the same unit. You cannot say that 2 yards and 3 feet make 5 *yards* or 5 *feet*, because 2 and 3 make 5; yet you may say that 2 *yards* and 3 *yards* make 5 *yards*, and that 2 *feet* and 3 *feet* make 5 *feet*. It would be absurd to try to measure a quantity of one kind with a unit which is a quantity of another kind; for example, to attempt to tell how many yards there are in a gallon, or how many bushels of corn there are in a barrel of wine.

(16.) All things which are true of some numbers of one unit are true of the same numbers of any other unit. Thus, 15 pebbles and 7 pebbles together make 22 pebbles; 15 acres and 7 acres together make 22 acres; and so on. From this we come to say that 15 and 7 make 22, meaning that 15 things of the same kind and 7 more, of the same kind as the first, together make 22 of that kind, whether the kind mentioned be pebbles, horsemen, acres of land, or any other. For these it is but necessary to say, once for all, that 15 and 7 make 22. Therefore, in future, on this part of the subject I shall cease to talk of any particular units, such as pebbles or acres, and speak of numbers only.

(17.) I will now repeat the principal things which have been mentioned in this chapter, with some additions.

I. Ten signs are used, one to stand for nothing, the rest for the first nine numbers. They are

0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The first of these is called a *cipher*.

II. Higher numbers have not signs for themselves, but signs are made for them; by placing the signs already mentioned by the side of each other, and agreeing that the first figure on the right hand shall keep the value which it has when it stands alone; that the second on the right hand shall mean ten times as many as it does when it stands alone; that the third figure shall mean one hundred times as many as it does when it stands alone; the fourth one thousand times as many; and so on.

III. The right hand figure is said to be in the *units' place*, the next to that in the *tens' place*, the third in the *hundreds' place*, and so on.

IV. When a number is itself an exact number of tens, hundreds, or thousands, &c., as many ciphers must be placed on the right of it as will bring the number into the place which is intended for it. The following are examples:

Fifty, or five tens	50
Seven hundred	700

Five hundred and twenty-eight thousand 528000

If it were not for the ciphers these numbers would be mistaken for 5, 7, and 528.

(18.) A number may be expressed in many different ways. Thus, the number 16785 is either made up of

1 ten-thous. 6 thous. 7 hunds. 8 tens and 5
or 16 thousands 78 tens and 5
or 1 ten-thousand 678 tens and 5
or 167 hundreds 8 tens and 5
or 1678 tens and 5, and so on.

(19.) EXERCISES.

I. Write down the signs for
Four hundred and seventy-six—
Two thousand and ninety-seven—
Sixty-four thousand three hundred and fifty—
Two millions seven hundred and four—
Five hundred and seventy-eight millions of millions.

II. Write at full length 53, 1805, 1830, 66707, 180917324, 667137721, 90976390.

III. What alteration is made in a number made up entirely of nines, such as 99999, by adding one to it?

IV. Show that a number which has five figures in it is greater than one which has four, though the first has none but small figures in it, and the second none but large ones. For example, that 10111 is greater than 9879.

(20.) You now see that the convenience of our method of numeration arises from a few simple signs being made to change their value as they change the column in which they are placed. The same convenience arises from counting in the same way all the articles which are used in every-day life. For example, we count money by dividing it into pounds, shillings, and pence, of which a shilling is 12 pence, and a pound 20 shillings, or 240 pence. We write a number of pounds, shillings, and pence in three columns, generally placing a point between each column. Thus, 263 pence would not be written as 263, but as £1 . 1 . 11, where £ shows the 1 in the first column is a pound. Here is a *system of numeration* in which a number in the second column on the right means 12 times as much as the same number in the first; and one in the third column is 20 times as great as the same in the second, or 240 times as great as the same in the first. In each of the tables of measures which you will hereafter meet with, you will see a separate system of numeration, but the methods of calculation for all will be the same.

(21.) In order to make the language of arithmetic shorter, some other signs are used. They are as follows:

I. $15 + 38$ means that 38 is to be added to 15, and is the same thing as 53.

II. $64 - 12$ means that 12 is to be taken away from 64, and is the same thing as 52.

III. 9×8 means that 9 is to be taken 8 times, and is the same thing as 72.

IV. $\frac{108}{6}$ means that 108 is to be divided by 6, or that you must find out how many sixes there are in 108; and is the same thing as 18.

V. When two numbers, or collections of numbers, with the foregoing signs, are the same, the sign $=$ is put between them: thus, that 7 and 5 make 12, is written in this way, $7+5=12$.

(22.) EXERCISES.

Read or write the following sentences in common language, and find out whether they are true or not.

$$1+2 \ldots\ldots\ldots = \frac{2\times 3}{2}$$

$$1+2+3 \ldots\ldots = \frac{3\times 4}{2}$$

$$1+2+3+4 \ldots = \frac{4\times 5}{2}$$

$$1+2+3+4+5 = \frac{5\times 6}{2}$$

and so on.

$$1+3 \ldots\ldots\ldots = 2\times 2$$

$$1+3+5 \ldots\ldots = 3\times 3$$

$$1+3+5+7 \ldots = 4\times 4$$

$$1+3+5+7+9 = 5\times 5$$

and so on.

SECTION II.

ADDITION AND SUBTRACTION.

(23.) There is no process in arithmetic which does not consist entirely in the increase or diminution of numbers. There is then nothing which might not be done with a number of pebbles. Probably at first nothing but pebbles, or the fingers, were used. Our word *calculation* is derived from the Latin word *calculus*, which means a pebble; and, long after better methods of counting were known, the Romans used the words *vocare ad calculos*, "to call it to pebbles," meaning, to reckon it accurately.

(24.) Shorter ways of counting have been invented, by which many calculations, which would require long and tedious reckoning, if pebbles were used, are made at once with very little trouble. The four great methods of counting are, Addition, Subtraction, Multiplication, and Division; of which, the last two are only ways of doing several of the first and second at once.

(25.) When one number is increased by another, the number which is as large as both the others together is called their *sum*. The process of finding the sum of two or more numbers is called ADDITION, and, as was said before, is denoted by placing a cross (+) between the numbers which are to be added together.

Suppose it required to find the sum of 1834 and 2799. In order to add these numbers, take them to pieces, dividing each into its units, tens, hundreds, and thousands:

1834 is 1 thous. 8 hund. 3 tens and 4,
2799 is 2 thous. 7 hund. 9 tens and 9.

Each of these numbers is thus broken up into four parts. If to each part of the first you add the part of the second which is under it, and then put together what you get from these additions, you will have added 1834 and 2799. In the first number are 4 units and in the second 9: these will, when the numbers are added together, contribute 13 units to the sum. Again, the 3 tens in the first and the 9 tens in the second will contribute 12 tens to the sum. The 8 hundreds in the first and the 7 hundreds in the second will add 15 hundreds to the sum; and the thousand in the first with the 2 thousands in the second will contribute 3 thousands to the sum; therefore the sum required is

3 thousands, 15 hundreds, 12 tens, and 13 units.

To make this into one number, you must recollect that

13 units are . . . . 1 ten and 3 units,
12 tens are . . . . 1 hund. and 2 tens,
15 hund. are 1 thous. and 5 hund.
3 thous. are 3 thous.

Now collect the numbers on the right hand side together, as was done before, and this will give, as the sum of 1834 and 2799,
4 thousands, 6 hundreds, 3 tens, and 3 units, which (17) is written 4633.

(26.) The former process, written with the signs of (21) is as follows:—

$$1834 = 1\times 1000 + 8\times 100 + 3\times 10 + 4$$
$$2799 = 2\times 1000 + 7\times 100 + 9\times 10 + 9$$

Therefore

$$1834 + 2799 = 3\times 1000 + 15\times 100 + 12\times 10 + 13$$

$$\begin{array}{rrl}
\text{But} & 13 = & \qquad\qquad\qquad 1\times 10+3 \\
12\times & 10 = & \qquad\quad 1\times 100+2\times 10 \\
15\times & 100 = & 1\times 1000+5\times 100 \\
3\times & 1000 = & 3\times 1000.\ \ \text{Therefore} \\
\multicolumn{2}{r}{1834+2799=} & 4\times 1000+6\times 100+3\times 10+3
\end{array}$$

$=4633.$

(27.) The same process is to be followed in all cases, but not at the same length. In order to be able to go through it, you must know how to add together the simple numbers. This can only be done by memory; and to help the memory you should make the following table three or four times for yourself:

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

The use of this table is as follows: Suppose you want to find the sum of 8 and 7. Look in the left hand column for either of them; 8 for example, and look in the top column for 7. On the same line as 8, and underneath 7, you find 15, their sum.

(28.) When this table has been thoroughly committed to memory, so that you can tell at once the sum of any two numbers, neither of which exceeds 9, you should exercise yourself in adding and subtracting two numbers, one of which is greater than 9 and the other less. You should write down a great number of such sentences as the following, which will exercise you at the same time in addition, and in the use of the signs mentioned in (21).

$12 + 6 = 18$ $22 + 6 = 28$ $19 + 8 = 27$
$54 + 9 = 63$ $56 + 7 = 63$ $22 + 8 = 30$
$100 - 9 = 91$ $27 - 8 = 19$ $44 - 6 = 38$, &c.

(29.) When the last two articles have been thoroughly studied, you will be able to find the sum of any numbers by the following process, which is the same as that in (25).

Rule I. Place the numbers in such a way that the units of all may be under one another.

II. Add together the units of all, and part the *whole* number thus obtained into units and tens. Thus, if 85 is the number, part it into 8 tens and 5 units; if 136 be the number, part it into 13 tens and 6 units (18).

III. Write down the units of this number under the units of the rest, and keep in memory the number of tens.

IV. Add together all the numbers in the column of tens, remembering to take in (or carry, as it is called) the tens, which you were told to recollect in III., and divide this number of tens into tens and hundreds. Thus, if 335 tens is the number obtained, part this into 33 hundreds and 5 tens.

V. Place the number of tens under the tens, and remember the number of hundreds.

VI. Proceed in this way through every column, and at the last column, instead of separating the number you obtain into two parts, write it all down before the rest.

Example. What is

1805 + 36 + 19727 + 3 + 1474 + 2008

$$\begin{array}{r}1805\\36\\19727\\3\\1474\\2008\\\hline 25053\end{array}$$

The addition of the units line, or 8 + 4 + 3 + 7 + 6 + 5, gives 33, that is, 3 tens and 3 units. Put 3 in the units' place, and add together the line of tens, taking in at the beginning the 3 tens which were created by the addition of the units' line. That is, find 3 + 0 + 7 + 2 + 3 + 0, which gives 15 for the number of tens; that is, 1 hundred and 5 tens. Add the line of hundreds together, taking care to add the 1 hundred which arose in the addition of the line of tens; that is, find 1 + 0 + 4 + 7 + 8, which gives exactly 20 hundreds, or 2 thousands and no hundreds. Put a cipher in the hundreds' units' place (because, if you do not, the next figure will be taken for hundreds instead of thousands), and add the figures in the thousands' line together, remembering the 2 thousands which arose from the hundreds' line; that is, find 2 + 2 + 1 + 9 + 1, which gives 15 thousands, or 1 ten thousand and 5 thousand. Write 5 under the line of thousands, and collect the figures in the line of tens of thousands, remembering the ten thousand which arose out of the thousands' line; that is, find 1 + 1 or 2 ten thousands. Write 2 under the ten thousands' line, and the operation is completed.

(30.) As an exercise in addition you may satisfy yourself that what I now say of the following square is correct. The numbers in every row, whether reckoned upright, or from right to left, or from corner to corner, when added together give the number 24156.

2016	4212	1656	3852	1296	3492	936	3132	576	2772	216
252	2052	4248	1692	3888	1332	3528	972	3168	612	2412
2448	288	2088	4284	1728	3924	1368	3564	1008	2808	648
684	2484	324	2124	4320	1764	3960	1404	3204	1044	2844
2880	720	2520	360	2160	4356	1800	3600	1440	3240	1080
1116	2916	756	2556	396	2196	3996	1836	3636	1476	3276
3312	1152	2952	792	2592	36	2232	4032	1872	3672	1512
1548	3348	1188	2988	432	2628	72	2268	4068	1908	3708
3744	1584	3384	828	3024	468	2664	108	2304	4104	1944
1980	3780	1224	3420	864	3060	504	2700	144	2340	4140
4176	1620	3816	1260	3456	900	3096	540	2736	180	2376

(31.) If two numbers are to be added together, it will not alter the sum, if you take away a part of one, provided you put on as much to the other. It is plain that you will not alter the whole number of a collection of pebbles in two baskets by taking any number out of one, and putting them in the other. Thus 15 + 7 is the same as 12 + 10, since 12 is 3 less than 15, and 10 is 3 more than 7. You may see that this was the principle upon which the whole of the process in (25) was conducted.

(32.) When one number is taken away from another, the number which is left is called the *difference* or *remainder*. The process of finding the difference is called SUBTRACTION.

(33.) The process of subtraction depends upon these two principles.

I. The difference of two numbers is not altered by adding a number to the first, if you add the same number to the second. Conceive two baskets with pebbles in them, in the first of which are 100 pebbles more than in the second. If I put 50 more pebbles into each of them, there are still only 100 more in the first than in the second. Therefore, in finding the difference of two numbers, if it should be convenient I may add any number I please to both of them, because, though I alter the numbers themselves by so doing, I do not alter their difference.

II. Since 6 exceeds 4 by 2,
and 3 exceeds 2 by 1,
and 12 exceeds 5 by 7,

6, 3, and 12 together, or 21, exceeds 4, 2, and 5 together, or 11, by 2, 1, and 7 together, or 10; the same thing may be said of any other numbers.

(34.) I want to find the difference of the numbers 57762 and 34631. Take these to pieces as in (25) and

57762 is 5 ten-th. 7 th. 7 hund. 6 tens and 2 units.
34631 is 3 ten-th. 4 th. 6 hund. 3 tens and 1 unit.

Now 2 units exceed 1 unit .... by 1 unit,
6 tens......... 3 tens ...... 3 tens,
7 hundreds.... 6 hundreds .. 1 hundred,
7 thousands ... 4 thousands .. 3 thousands,
5 ten-thousands 3 ten-thous. .. 2 ten-thous.

Therefore, by (33. Principle II.) all the first column *together* exceeds all the second column by all the third column, that is, by

2 ten thous. 3 thous. 1 hundred, 3 tens, and 1 unit, which is 23131. Therefore the difference of 57762 and 34631 is 23131, or 57762 – 34631 = 23131.

(35.) Suppose I want to find the difference between 61274 and 39628. Write them at length, and

61274 is 6 ten-th. 1 th. 2 hund. 7 tens and 4 units,
39628 is 3 ten-th. 9 th. 6 hund. 2 tens and 8 units.

If we attempt to do the same as in the last article, there is a difficulty immediately, since 8, being greater than 4, cannot be taken from it. But from (33) it appears that we shall not alter the difference of two numbers if we add the same number to *both* of them. Add ten to the first number, that is, let there be 14 units instead of four, and add ten also to the second number, but instead of adding ten to the number of units, add one to the number of tens, which is the same thing. The numbers will then stand thus,

6 ten-thou. 1 thou. 2 hund. 7 tens and 14 *units*[*]
3 ten-thou. 9 thou. 6 hund. 3 *tens* and 4 units.

You now see that the units and tens in the lower can be subtracted from those in the upper line, but that the hundreds cannot. To remedy this, add one thousand or 10 hundred to both numbers, which will not alter their difference, and recollect to increase the hundreds in the upper line by 10, and the thousands in the lower line by 1, which is the same thing. And since the thousands in the lower cannot be subtracted from the thousands in the upper line, add 1 ten thousand or 10 thousand to both numbers, and increase the thousands in the upper line by 10, and the ten thousands in the lower line by 1, which is the same thing; and at the close the numbers which we get will be

6 ten-thous. 11 *thous.* 12 *hund.* 7 tens and 14 *units*
4 *ten-thous.* 10 *thous.* 6 hund. 3 *tens* and 8 units.

These numbers are not, it is true, the same as those given at the beginning of this article, but their difference is the same by (33). With the last mentioned numbers proceed in the same way as in (34), which will give, as their difference,
2 ten-thousands, 1 thousand, 6 hundreds, 4 tens, and 6 units, which is 21646.

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[*] Those numbers which have been altered are put in italics.

(36.) From this we can deduce the following rules for subtraction:

I. Write the number which is *to be subtracted* (which is of course the lesser of the two, and which is called the *subtrahend*) under the other, so that its units shall fall under the units of the other, and so on.

II. Subtract each figure of the lower line from the one above it, if that can be done. Where that cannot be done, add ten to the upper figure, and then subtract the lower figure, but recollect in this case always to increase the next figure in the lower line by 1, before you begin to subtract it from the upper one.

(37.) If there should not be as many figures in the lower line as in the upper one, proceed as if there were as many ciphers at the beginning of the lower line as will make the number of figures equal. You do not alter a number by placing ciphers at the beginning of it. For example, 00818 is the same number as 818, for it means

0 ten-thous. 0 thous. 8 hunds. 1 ten and 8 units;

the two first numbers are nothing, and the rest is

8 hundreds, 1 ten, and 8 units, or 818.

The second does not differ from the first, except in its being said that there are no thousands and no tens of thousands in the number, which may be known without their being mentioned at all. You may ask, perhaps, why this does not apply to a cipher placed in the middle of a number, or at the right of it, as, for example, in 28007 and 39700.

But you must recollect, that if it were not for the two ciphers in the first, the 8 would be taken for 8 tens, instead of 8 thousands; and if it were not for the ciphers in the second, the 7 would be taken for 7 units, instead of 7 hundreds.

(38.) EXAMPLE.

What is the difference between 3708291640030174
and 30813649276188
Difference 3677477990753986

EXERCISES.

I. What is 18337 + 149263200 − 6472902
and 1000 − 464+3279 − 646?

II. Subtract
64 + 76 + 144 − 18 from 33 − 2 + 100037.

III. What shorter rule might be made for subtraction when all the figures in the upper line are ciphers except the first? for example, in finding
10000000 − 2731634.

IV. Find 18362 + 2469 and 18362 − 2469, add the second result to the first, and then subtract 18362; subtract the second from the first, and then subtract 2469.

V. There are four places on the same line in the order A, B, C, and D. From A to D it is 1463 miles; from A to C it is 728 miles; and from B to D it is 1317 miles. How far is it from A to B, from B to C, and from C to D?

SECTION III.

MULTIPLICATION.

(39.) I have said that all questions in arithmetic require nothing but addition and subtraction. I do not mean by this that no rule should ever be used except those given in the last section, but that all other rules only show shorter ways of finding what might be found, if we pleased, by the rules in the last section. Even the last two rules themselves are only short and convenient ways of doing what may be done with a number of pebbles or counters. So that there is nothing in arithmetic but what may be done with these pebbles or counters.

(40.) I want to know the sum of five seventeens, or I ask the following question—There are 5 heaps of pebbles, and seventeen pebbles in each heap; how many are there in all? Write five seventeens in a column, and make the addition, which gives 85. In this case 85 is called the *product* of 5 and 17, and the process of finding the *product* is called MULTIPLICATION, which is nothing more than the addition of a number of the same quantities. Here 17 is called the *multiplicand*, and 5 is called the *multiplier*.

17
17
17
17
17
—
85

(41.) If no question harder than this were ever proposed, there would be no occasion for a shorter way than the one which I followed. But if there were 1367 heaps of pebbles and 429 in each heap, the whole number is then 1367 times 429, or 429 multiplied by 1367. I should have to write 429 1367 times, and then to make an addition of enormous difficulty. To avoid this, a shorter rule is necessary, which I now proceed to explain.

(42.) The student must first make himself acquainted with the products of all numbers as far as 10 times 10, by means of the following table *, which must be committed to memory.

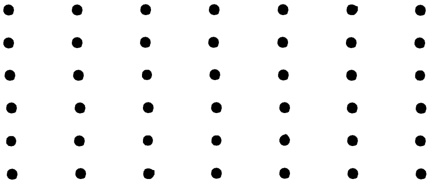
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

---

* As it is usual to learn the products of numbers as far as 12 times 12, I have extended the table thus far. In my

If from this table you wish to know what is 7 times 6, look in the first upright column on the left for either of them; 6 for example. Proceed to the right until you come into the column marked 7 at the top. You there find 42, which is the product of 6 and 7.

(43.) You may find, in this way, either 6 times 7, or 7 times 6, and for both you find 42. That is, six sevens is the same number as seven sixes. This may be shown as follows: Place seven counters in a line, and repeat that line in all six times, thus:



The number of counters in the whole is 6 times 7, or six sevens, if I reckon the rows from the top to the bottom; but if I count the rows that stand side by side, I find seven of them, and six in each row, the whole number of which is 7 times 6, or seven sixes. And the whole number is 42, whichever way I count. The same method may be applied to any other two numbers. If the signs of (21.) were used, it would be said that 7 × 6 = 6 × 7.

(44.) To take any quantity a number of times,

opinion, all pupils who show a tolerable capacity should commit the products to memory, as far as 20 times 20.

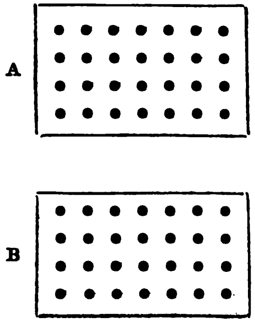
it will be enough to take every one of its parts the same number of times. Thus a sack of corn will be increased fifty-fold, if each bushel which it contains be replaced by 50 bushels. A country will be doubled by doubling every acre of land, or every county which it contains. Simple as this may appear, it is necessary to state it, because it is one of the principles on which the rule of multiplication depends.

(45.) In order to multiply by any number, you may multiply separately by any parts into which you choose to divide that number, and add the results. For example, 4 and 2 make 6—To multiply 7 by 6, first multiply 7 by 4, and then by 2, and add the products. This will give 42, which is the product of 7 and 6. Again, since 57 is made up of 32 and 25, 50 times 57 is made up of 32 times 57, and 25 times 57, and so on. If the signs were used, these would be written thus:

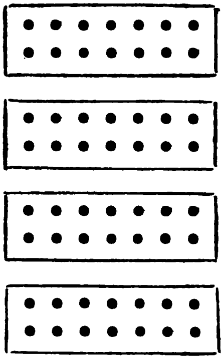
$$7 \times 6 = 7 \times 4 + 7 \times 2$$
$$50 \times 57 = 50 \times 32 + 50 \times 35$$

(46.) There is another way in which two numbers may be multiplied together. Since 8 is 4 times 2, 7 times 8 may be made by multiplying 7 and 4, and then multiplying that *product* by 2. To show this, place 7 counters in a line, and repeat that line in all 8 times, as in figures I. and II.

I.



II.



The number of counters in all is 8 times 7, or 56. But (as in fig. I.) enclose each four rows in oblong figures, such as A and B. The number in each oblong is 4 times 7, or 28, and there are two of those oblongs; so that in the whole the number of counters is twice 28, or 28 × 2, or 7 first multiplied by 4, and that product multiplied by 2. In figure II., it is shown that 7 multiplied by 8 is also 7 first multiplied by 2, and that product multiplied by 4. The same method may be applied to other numbers. Thus, since 80 is 8 times 10, 256 times 80 is 256 multiplied by 8, and that product multiplied by 10. If we use the signs, the foregoing assertions are made thus:

$$7 \times 8 = 7 \times 4 \times 2 = 7 \times 2 \times 4$$

$$256 \times 80 = 256 \times 8 \times 10 = 256 \times 10 \times 8.$$

EXERCISES.

Show that $2 \times 3 \times 4 \times 5 = 2 \times 4 \times 3 \times 5 = 5 \times 4 \times 2 \times 3$, &c.

Show that $18 \times 100 = 18 \times 57 + 18 \times 43$.

(47.) In order to multiply by 10, annex a cipher to the right hand of the multiplicand. Thus 10 times 2356 is 23560. To show this, write 2356 at length, which is

2 thousands, 3 hundreds, 5 tens, and 6 units.

Take each of these parts ten times, which, by (44), is the same as multiplying the whole number by 10, and it will then become

2 tens of thou. 3 tens of hun. 5 tens of tens, and 6 tens, which is, 2 ten-thous. 3 thous. 5 hun. and 6 tens.

This must be written 23560, because 6 is not to be 6 units, but 6 tens. Therefore $2356 \times 10 = 23560$.

In the same way you may show that in order to multiply by 100 you must affix 2 ciphers to the right: to multiply by 1000 you must affix three ciphers, and so on. The rule will be best caught from the following table.

$13 \times 10 = 130$

$13 \times 100 = 1300$

$13 \times 1000 = 13000$

$13 \times 10000 = 130000$

$142 \times 1000 = 142000$

$23700 \times 10 = 237000$

$3040 \times 1000 = 3040000$

$10000 \times 100000 = 1000000000$

(48.) I now show how to multiply by one of the numbers 2, 3, 4, 5, 6, 7, 8, or 9. I do not include 1, because multiplying by 1, or taking the number once, is what is meant by simply writing down the number. I want to multiply 1368 by 8. Write the first number at full length, which is

1 thousand, 3 hundreds, 6 tens, and 8 units.

To multiply this by 8, multiply each of these parts by 8 (42) and (44), which will give

8 thousands, 24 hundreds, 48 tens, and 64 units.

Now 64	units are written thus .	64
48	tens	480
24	hundreds	2400
8	thousands	8000

Add these together, which gives 10944 as the product of 1368 and 8, or $1368 \times 8 = 10944$. By working a few examples in this way, you will see the following rule.

(49.) I. Multiply the first figure of the multiplicand by the multiplier, write down the right hand figure, and reserve the tens.

II. Do the same with the second figure of the multiplicand, and add to the product the number of tens from the first, put down the units figure of this and reserve the tens.

III. Proceed in this way till you come to the last figure, and then write down the whole number obtained from that figure.

IV. If there be a cipher in the multiplicand, treat it as if it were a number, observing that $0 \times 1 = 0$, $0 \times 2 = 0$, &c.

(50.) In a similar way a number can be multiplied by a figure which is accompanied by ciphers, as, for example, 8000. For 8000 is 8×1000, and therefore (46) you must first multiply by 8 and then by 1000, which last operation (47) is done by placing 3 ciphers on the right. Hence the rule in this case is—Multiply by the simple number, and place the number of ciphers which follow it at the right of the product.

EXAMPLE.

Multiply 1679423800872
by 60000

100765428052320000

(51.) **EXERCISES.**

What is 1007360 × 7
123456789 × 9 + 10 and 123 × 9 + 4?
What is 136 × 3 + 129 × 4 + 147 × 8 + 27 × 3000
and 30 × 40 + 180 × 1700 − 26 × 50?

An army is made up of 33 regiments of infantry, each containing 800 men; 14 of cavalry, each containing 600 men; and two of artillery, each containing 300 men. The enemy has 6 more regiments of infantry, each containing 100 more men; 3 more regiments of cavalry, each containing 100 men less; and 4 corps of artillery of the same magnitude as those of the first: two regiments of cavalry and one of infantry desert to the enemy. How many men has the second army more than the first?

(52.) Suppose it be required to multiply 23707 by 4567. Since 4567 is made up of 4000, 500, 60, and 7; by (45) we must multiply 23707 by each of these, and add the products.

Now (48)	23707 × 7	is	165949
(50)	23707 × 60	is	1422420
	23707 × 500	is	11853500
	23707 × 4000	is	94828000
The sum of these is			108269869

which is the product required.

It will do as well if, instead of writing the ciphers at the end of each line, we keep the other figures in their places without them. Now, if we take away the ciphers, the second line is one place to the right of the first, the third one place to the right of the second, and so on. Write the multiplier and the multiplicand over these lines, and the process will stand thus:—

```
    23707
     4567
   ------
   165949
  142242
 118535
 94828
---------
108269869
```

(53.) There is only one case to be noticed, and that is where there is a cipher in the middle of the multiplier. The following example will show that in this case nothing more is necessary than to keep the first figure of each line in the column under the figure of the multiplier from which that line arises. Suppose it required to multiply 365 by 101001. The multiplier is made up of 100000, 1000, and 1. Proceed as before, and

	365 × 1	is	365
(50)	365 × 1000	is	365000
	365 × 100000	is	36500000
	The sum of which is		36865365

and the whole process with the ciphers struck off is

$$\begin{array}{r}
365 \\
101001 \\
\hline
365 \\
365 \\
365 \\
\hline
36865365
\end{array}$$

(54.) The following is the rule in all cases:

I. Place the multiplier under the multiplicand, so that the units of one may be under those of the other.

II. Multiply the whole multiplicand by each figure of the multiplier (50), and place the unit of each line, in the column under the figure of the multiplier from which it came.

III. Add together the lines obtained by II. column by column.

(55.) When the multiplier or multiplicand, or both, have ciphers on the right hand, multiply the two together without the ciphers, and then place on the right of the product all the ciphers that are on the right both of the multiplier and multiplicand. For example, what is 3200 × 13000?—For 3200 is 32 × 100, or one hundred times as great as 32. Again, 32 × 13000 is 32 × 13, with three ciphers affixed, that is, 416, with three ciphers affixed, or 416000. But the product required must be 100 times as great as this, or must have two ciphers affixed. It is therefore 41600000, having as many ciphers as are in both multiplier and multiplicand.

(56.) EXERCISES.

What is 43730×2729, 123×321, 5676307×302018, and 1247×1247000?

Find $13 \times 13 + 14 \times 14 - 12 \times 12$.

Show that

I. $169 \times 169 = 119 \times 119 + 120 \times 120$.

II. $130 \times 130 - 26 \times 26 = 156 \times 104$.

III. $18 \times 18 \times 18 - 17 \times 17 \times 17 = 18 \times 18 + 17 \times 18 + 17 \times 17$.

IV. $57 \times 57 = 50 \times 50 + 2 \times 50 \times 7 + 7 \times 7$.

V. $12 \times 12 \times 12 = 8 \times 8 \times 8 + 3 \times 8 \times 8 \times 4 + 3 \times 8 \times 4 \times 4 + 4 \times 4 \times 4$.

VI. $187 \times 187 - 186 \times 186 = 187 + 186$.

Find $174 \times 174 \times 3{\cdot}14159$, and $2 \times 663 \times 31459$.

SECTION IV.

DIVISION.

(57.) Suppose I ask whether 156 can be divided into a number of parts each of which is 13, or how many thirteens 156 contains; I propose a question the solution of which is called DIVISION. In this case 156 is called the *dividend*, 13 the *divisor*, and the number of parts required is the *quotient;* and when I find the quotient I am said to divide 156 by 13.

(58.) The simplest method of doing this is to subtract 13 from 156, and then to subtract 13 from the remainder, and so on, or, in common language, to *tell off* 156 by thirteens. Do this and mark one for every subtraction that is made to remind you that each subtraction takes 13 once from 156, which operations will stand as follows—

$$\begin{array}{rr}
156 & \\
13 & 1 \\ \hline
143 & \\
13 & 1 \\ \hline
130 & \\
13 & 1 \\ \hline
117 & \\
13 & 1 \\ \hline
104 & \\
13 & 1 \\ \hline
91 & \\
13 & 1 \\ \hline
78 & \\
13 & 1 \\ \hline
65 & \\
13 & 1 \\ \hline
52 & \\
13 & 1 \\ \hline
39 & \\
13 & 1 \\ \hline
26 & \\
13 & 1 \\ \hline
13 & \\
13 & 1 \\ \hline
0 &
\end{array}$$

Begin by subtracting 13 from 156, which leaves 143. Subtract 13 from 143, which leaves 130, and so on. At last 13 only remains, from which when 13 is subtracted there remains nothing. Upon counting the number of times which you have subtracted 13, you find that this number is 12, or 156 contains twelve thirteens, or contains 13 twelve times.

This method is the most simple possible, and might be done with pebbles. Of these you would first count 156. You would then take thirteen from the heap, and put them into one heap by themselves. You would then take another 13 from the heap, and place them in another heap by themselves, and so on, until there were none left. You would then count the number of heaps, which you would find to be 12.

(59.) Division is the opposite of multiplication. In multiplication you have a number of heaps, with the same number of pebbles in each, and you want to know how many *pebbles* there are in all. In division, you know how many there are in all, and how many there are to be in each heap, and you want to know how many *heaps* there are.

(60.) In the last example, a number was taken which contains an exact number of thirteens. But this does not happen with every number. Take, for example, 159. Follow the process of (58), and it will appear that after having subtracted 13 twelve times, there remains 3, from which 13 cannot be subtracted. We say then that 159 contains twelve thirteens and 3 *over*, or that 159 when divided by 13 gives a *quotient* 12, and a *remainder* 3. If we use signs, $159 = 13 \times 12 + 3$.

EXERCISES.

$146 = 24 \times 6 + 2$, or 146 contains six twenty-fours and 2 over.

$146 = 6 \times 24 + 2$, or 146 contains twenty-four sixes and 2 over.

$3000 = 42 \times 7 + 6$, or 3000 contains seven forty-twos and 6 over.

$39624 = 7277 \times 5 + 3239$.

(61.) If I divide 156 into several parts, and find how often 13 is contained in each of them; it is plain that 156 contains 13 as often as all its parts together. For example, 156 is made up of 91, 39, and 26. Of these,

91 contains 13, 7 times,
39 contains 13, 3 times,
26 contains 13, 2 times;

therefore, $91 + 39 + 26$ contains 13, $7 + 3 + 2$ times, or 12 times.

Again, 156 is made up of 100, 50, and 6.

Now, 100 contains 13, 7 times and 9 over,
50 contains 13, 3 times and 11 over,
6 contains 13, 0 times* and 6 over.

Therefore, 100 + 50 + 6 contains 13, 7 + 3 + 0 times and 9 + 11 + 6 over; or 156 contains 13, 10 times and 26 over. But 26 is itself 2 thirteens; therefore, 156 contains 10 thirteens, and 2 thirteens, or 12 thirteens.

(62.) In the first example, I did not take 13 away more than once at a time, which was done that the method might be as simple as possible. But if I know what is twice 13, 3 times 13, &c., I can take away as many thirteens at a time as I please, if I take care to mark at each step how many I take away. For example, take away 13 ten times at once from 156, that is, take away 130, and afterwards take away 13 twice, or take away 26, and the process is as follows:

$$\begin{array}{r l} 156 & \\ 130 & 10 \text{ times } 13 \\ \hline 26 & \\ 26 & 2 \text{ times } 13 \\ \hline 0 & \end{array}$$

Therefore, 156 contains 13, 10 + 2, or 12 times.

---

* To speak always in the same way, instead of saying that 6 does not contain 13, I say that it contains it 0 times and 6 over, which is merely saying that 6 is 6 more than nothing.

Again, to divide 3096 by 18.

$$\begin{array}{rl} 3096 & \\ 1800 & 100 \text{ times } 18 \\ \cline{1-1} 1296 & \\ 900 & 50 \text{ times } 18 \\ \cline{1-1} 396 & \\ 360 & 20 \text{ times } 18 \\ \cline{1-1} 36 & \\ 36 & 2 \text{ times } 18 \\ \cline{1-1} 0 & \end{array}$$

Therefore, 3096 contains 18, $100+50+20+2$, or 172 times.

(63.) You will now understand the following sentences, and be able to make similar assertions of other numbers.

450 is six times 75; it therefore contains 5 six times as often as 75 contains it.

135 contains 3 more than 26 times: therefore, twice
135 contains 3 more than 52 times, 52 being twice 26; ten times
135 contains 3 more than 260 times.
472 contains 18 more than 21 times,
4720 contains 18 more than 210 times,
47200 contains 18 more than 2100 times.

32 contains 12 more than 2 times, and less than 3 times,

320 contains 12 more than 20 times, and less than 30 times

3200 contains 12 more than 200 times, and less than 300 times—and so on.

(64.) The foregoing articles contain the principles of division. The question now is to apply them in the shortest and most convenient way. Suppose it required to divide 4068 by 18, or to find $\frac{4068}{18}$ (21).

If we divide 4068 into any number of parts, we may, by the process followed in (61), find how many times 18 is contained in each of these parts, and from thence how many times it is contained in the whole. Now, what separation of 4068 into parts will be most convenient? Observe that 4, the first figure of 4068, does not contain 18; but that 40, the first and second figures together, *does contain* 18 *more than twice, but less than three times**. But 4068 (18) is made up of 40 hundreds, and 68; of which, 40 hundreds (63) contains 18 more than 200 times, and less than 300 times. Therefore, 4068 also contains more than 200 times 18, since it must contain 18 more times than 4000 does. It also contains 18 less than 300 times, because 300 times 18 is 5400, a greater number than 4068.

---

* If you have any doubt as to this expression recollect that it means " contains more than two eighteens, but not so much as three."

Subtract 18, 200 times from 4068; that is, subtract 3600, and there remains 468. Therefore, 4068 contains 18, 200 times, and as many times as 468 contains it.

It remains, then, to find how many times 468 contains 18. Proceed exactly as before. Observe that 46 contains 18 more than twice, and less than 3 times; therefore, 460 contains it more than 20, and less than 30 times (63); as does also 468. Subtract 18, 20 times from 468, that is, subtract 360, the remainder is 108. Therefore, 468 contains 18, 20 times, and as many more as 108 contains it. Now 108 is found to contain 18, 6 times exactly; therefore, 468 contains it 20 + 6 times, and 4068 contains it 200 + 20 + 6 times, or 226 times. If we write down the process that has been followed, without any explanation, putting the divisor, dividend, and quotient, in a line separated by brackets, it will stand thus—

$$
\begin{array}{rrl}
18) & 4068 & (200+20+6 \\
 & \underline{3600} & \\
 & 468 & \\
 & \underline{360} & \\
 & 108 & \\
 & \underline{108} & \\
 & 0 &
\end{array}
$$

Let it be required to divide 36326599 by 1342.

1342)36326599(20000 + 7000 + 60 + 9
26840000

9486599
9394000

92599
80520

12079
12078

1

As in the previous example, 36326599 is separated into 36320000 and 6599; the first four figures 3632 being separated from the rest, because it takes four figures from the left of the dividend to make a number which is greater than the divisor. Again, 36320000 is found to contain 1342 more than 20000, and less than 30000 times; and 1342 × 20000 is subtracted from the dividend, after which the remainder is 9486599. The same operation is repeated again and again; and the result is found to be, that there is a quotient 20000 + 7000 + 60 + 9, or 27069, and a remainder 1.

Before you proceed, you should now repeat the foregoing article at length in the solution of the following questions: What are

$$\frac{10093874}{3207}, \frac{66779922}{114433}, \text{ and } \frac{27182818}{13352}?$$

(65.) In the examples of the last article you will observe, 1st, that it is useless to write down the ciphers which are on the right of each subtrahend, provided that without them you keep each of the other figures in its proper place: 2d, that it is useless to put down the right hand figures of the dividend so long as they fall over ciphers, because they do not begin to have any share in the making of the quotient until, by continuing the process, they cease to have ciphers under them: 3d, that the quotient is only a number written at length, instead of the usual way. For example, the first quotient is 200+20+6, or 226: the second is 20000+7000 + 60 + 9, or 27069. Strike out, therefore, all the ciphers and the numbers which come above them, except those in the first line, and put the quotient in one line, and the two examples of the last article will stand thus:

18)4068(226	1342)36326599(27069
36	2684
46	9486
36	9394
108	9259
108	8052
0	12079
	12078
	1

(66.) Hence the following rule is deduced:

I. Write the divisor and dividend in one line, and place brackets on each side of the dividend.

II. Take off from the left hand of the dividend the least number of figures which together make a number greater than the divisor; find what number of times the divisor is contained in these, and write this number as the first figure of the quotient.

III. Multiply the divisor by the last-mentioned figure, and subtract the product from the number which was taken off at the left of the dividend.

IV. On the right of the remainder place the figure of the dividend which comes next after those already separated in II.: if the remainder thus increased be greater than the divisor, find how many times the divisor is contained in it; put this number at the right of the first figure of the quotient, and repeat the process: if not, annex the next figure of the dividend, and the next, and so on until it is greater; but remember to place a cipher in the quotient for every figure of the dividend which you are obliged to take, except the first.

V. Proceed in this way until all the figures of the dividend are exhausted.

(67.) This process may be made more simple when the divisor is not greater than 12, if you have sufficient knowledge of the multiplication table (42). For example, I want to divide 132976 by 4. At full length the process stands thus:

$$
\begin{array}{l}
4)132976(33244 \\
12 \\
\hline
12 \\
12 \\
\hline
9 \\
8 \\
\hline
17 \\
16 \\
\hline
16 \\
16 \\
\hline
0
\end{array}
$$

But you will recollect, without the necessity of writing it down, that 13 contains 4 three times with a remainder 1; this 1 you will place before 2, the next figure of the dividend, and you know that 12 contains 4, 3 times exactly, and so on. It will be more convenient to write down the quotient thus:

$$
\begin{array}{r}
4)132976 \\
\hline
33244
\end{array}
$$

(68.) Where the divisor is unity followed by ciphers, the rule becomes extremely simple, as you will see by the following examples:

```
10)2717316                   100)33429(334
   -------                       300
    271731  and rem. 6           ---
                                  342
                                  300
                                  ---
                                   429
                                   400
                                   ---
                                    29
```

This is then the rule:—Cut off as many figures from the right hand of the dividend as there are ciphers. These figures will be the remainder, and the rest of the dividend will be the quotient.

Or we may prove these results thus: from (18) 2717316 is 271731 tens and 6; of which the first contains 10, 271731 times, and the second not at all; the quotient is therefore 271731, and the remainder 6 (60). Again (18) 33429 is 334 hundreds and 29; of which the first contains 100, 334 times, and the second not at all: the quotient is therefore 334, and the remainder 29.

(69.) The following examples will show how the rule may be shortened when there are ciphers in the divisor. With each example is placed another containing the same process, with all unnecessary figures removed; and from the comparison of the two, the rule at the end of this article is derived.

I. 1782000)6424700000(3605	1782)6424700(3605
5346000	5346
10787000	10787
10692000	10692
9500000	9500
8910000	8910
590000	590000

II. 12300000)42176189300(3428	123)421761(3428
36900000	369
52761893	527
49200000	492
35618930	356
24600000	246
110189300	1101
98400000	984
11789300	11789300

The rule then is:—Strike out as many *figures* from the right of the dividend as there are *ciphers* at the right of the divisor. Strike out all the ciphers from the divisor, and divide in the usual way; but at the end of the process, place on the right of the remainder all those figures which were struck out of the dividend.

(70.) EXERCISES.

Find $\frac{1372000946}{6437900}$, $\frac{12431731}{6253}$, and $\frac{21719322}{118}$.

Divide 137000001 by 23600, and 2179366 by 18307.

Show that

I. $\frac{100\times100\times100-43\times43\times43}{100-43} = 100 \times 100 \ldots\ldots$
$+100\times43+43\times43.$

II. $\frac{100\times100\times100+43\times43\times43}{100\times43} = 100 \times 100 \ldots\ldots$
$-100\times43+43\times43.$

III. $\frac{76\times76+2\times76\times52+52\times52}{76+52} = 76 + 52.$

IV. $1 + 12 + 12 \times 12 + 12 \times 12 \times 12 \ldots$
$= \frac{12\times12\times12\times12-1}{12-1}.$

What is the nearest number to 1376429 which can be divided by 36300 without remainder?

If 36 oxen can eat 216 acres of grass in one year, and if a sheep eat half as much as an ox, how long will it take 49 oxen and 136 sheep together to eat 17750 acres?

(71.) Take any two numbers, one of which divides the other without remainder; for example, 32 and 4. Multiply both these numbers by any other number; for example, 6. The products will be 192 and 24. Now 192 contains 24 just as often as 32 contains 4. Suppose 6 baskets, each containing 32 pebbles, the whole number of which will be 192. Take 4 from one basket, time after time, until that basket is empty. It is plain that if, instead of taking 4 from that basket, I take 4 from each, the whole six will be emptied together; that is, 6 times 32 contains 6 times 4 just as often as 32 contains 4. The same reasoning applies to other numbers, and therefore *we do not alter the quotient if we multiply the dividend and divisor by the same number.*

(72.) Again, suppose that 200 is to be divided by 50. Divide both the dividend and divisor by the same number; for example, 5. Then 200 is 5 times 40, and 50 is 5 times 10. But by (71) 40 divided by 10 gives the same quotient as 5 times 40 divided by 5 times 10, and therefore *the quotient of two numbers is not altered by dividing both the dividend and divisor by the same number.*

(73.) From (46) if a number is multiplied successively by two others, it is multiplied by their product. Thus 27, first multiplied by 5, and the product multiplied by 3, is the same as 27 multiplied by 5 times 3, or 15. Also, if a number be divided by any number, and the quotient be divided by another, it is the same as if the first number had been divided by the product of the other two. For example, divide 288 by 4, the quotient of which is 72. Divide 72 by 3, the quotient of which is 24. Then, since, by (71), 72 divided by 3 leaves the same quotient as 4 times 72 divided by 4 times 3, that is, as 288 divided by 12, and each quotient is 24; therefore, if 288 be divided by 4, and the quotient divided by 3, the result is the same as if 288 had been divided by 4 times 3, or 12. The same reasoning applies to other numbers.

(74.) The following rules will be better understood by stating them in an example. If 32 be multiplied by 24, and divided by 6, the result is the same as if 32 had been multiplied by the quotient of 24 divided by 6, that is, by 4. For, since 24 is 6 times 4, 32 multiplied by 24 is 32 × 6 × 4, or 32 × 4 × 6 (46), which is 6 times 32 × 4, or 6 times 128. Now 6 is contained in 6 times 128, 128 times, or 32 × 4 times; that is, it makes no difference whether 32 is multiplied by 24 and the product divided by 6, or whether 32 is at once multiplied by the quotient of 24 divided by 6.

If we use the signs, this is expressed thus:

$$\frac{32 \times 24}{6} = 32 + \frac{24}{6}.$$

Again, if 48 be multiplied by 4, and that product be divided by 24, it is the same thing as if 48 were divided at once by the quotient of 24 divided by 4, that is, by 6. For 48 is 8 × 6, and 48 times 4 is 8 × 6 × 4, which is 8 × 24 (46). If we divide this product by 24, the quotient is 8, which is the same quotient as arises from dividing 48 at once by 6, the quotient of 24 and 4. If we use the signs, this is expressed thus: $\frac{48 \times 4}{24} = \frac{48}{\frac{24}{4}}$.

(75.) When one number divides another without leaving any remainder, or is contained an exact number of times in it, it is said to be a *measure* of that number, or to *measure* it. Thus 4 is a measure of 136, or measures 136; but it does not measure 137. The reason for using the word measure is this: Suppose you have a rod 4 feet long, with nothing marked upon it, with which you want to measure some length; for example, the length of a street. If that street should happen to be 136 feet in length, you will be able to *measure* it with that rod, because, since 136 contains 4 34 times, you will find that the street is exactly 34 times the length of the rod. But if the street should happen to be 137 feet long, you cannot measure it with that rod; for when you have measured 34 of the rods, you will find a remainder, whose length you cannot tell without some shorter measure. Hence 4 is said to measure 136, but not to measure 137. A measure, then, is a divisor which leaves no remainder.

(76.) When one number is a measure of two others, it is called a *common measure* of the two. Thus 15 is a common measure of 180 and 75. Two numbers may have several common measures. For example, 360 and 168 have the common measures 2, 3, 4, 6, 24, and several others. Now this question may be asked,—Of all the common measures of 360 and 168, which is the greatest? The answer to this question is derived from a rule of arithmetic, called the rule for finding the GREATEST COMMON MEASURE, to which we shall shortly come.

(77.) If one quantity measures two others, it measures their sum and difference. Thus 7 measures 21 and 56. It therefore measures $56+21$ and $56-21$, or 77 and 35. This is only another way of saying what was said in (61).

(78.) If one number measures a second, it measures every number which the second measures. Thus 5 measures 15, and 15 measures 30, 45, 60, 75, &c.; all which numbers are measured by 5. It is plain that if

15 contains 5 3 times,
30 or 15 + 15 contains 5 3 + 3 times, or 6 times,
45 or 15 + 15 + 15 contains 5 3 + 3 + 3 or 9 times,
and so on.

(79.) Every number which measures both the dividend and divisor measures the remainder also. To show this, divide 360 by 112. The quotient is 3, and the remainder 24, that is (60), 360 is three times 112 and 24, or $360 = 112 \times 3 + 24$. From this it follows that 24 is the difference between 360 and 3 times 112, or $24 = 360 - 112 \times 3$. Take any number which measures both 360 and 112; for example, 4. Then

4 measures 360,
4 measures 112, and therefore (78) measures 112×3, or $112 + 112 + 112$.

Therefore (77) it measures $360 - 112 \times 3$, which is the remainder 24. The same reasoning may be applied to all other measures of 360 and 112; and the result is, that every quantity which measures both the dividend and divisor also measures the remainder. Hence every *common measure* of a dividend and divisor is also a *common measure* of the divisor and remainder.

(80.) There is no common measure of the remainder and divisor which is not also a common measure of the dividend and divisor. Take the same example, and recollect that $360 = 112 \times 3 + 24$.

Take any common measure of the remainder 24 and the divisor 112; for example, 8. Then

8 measures 24;

and 8 measures 112, and therefore (78) measures 112 × 2.

Therefore (77) 8 measures 112 × 3 + 24, or measures the dividend 360. Then every common measure of the remainder and divisor is also a common measure of the divisor and dividend, or there is no common measure of the remainder and divisor which is not also a common measure of the divisor and dividend.

(81.) I. It is proved in (79) that the remainder and divisor have all the common measures which are in the dividend and divisor.

II. It is proved in (80) that they have no others.

It therefore follows that the greatest of the common measures of the first two is the greatest of those of the second two, which shows how to find the greatest common measure of any two numbers*.

(82.) Take the preceding example, and let it be required to find the g. c. m. of 360 and 112, and observe that

360 divided by 112 gives the remainder 24,
112 divided by 24 gives the remainder 16,
24 divided by 16 gives the remainder 8,
16 divided by 8 gives no remainder.

Now since 8 divides 16 without remainder, and since it also divides itself without remainder, 8 is the g. c. m. of 8 and 16, because it is impossible to divide 8 by any number greater than 8; so that, even if 16 had a greater measure than 8, it could not be *common* to 16 and 8.

---

* For shortness I will abbreviate the words *greatest common measure* into their initial letters, g. c. m.

Therefore 8 is g. c. m. of 16 and 8.
(81) g. c. m. of 16 and 8 is g. c. m. of 24 and 16
g. c. m. of 24 and 16 is g. c. m. of 112 and 24
g. c. m. of 112 and 24 is g. c. m. of 360 and 112.
Therefore 8 is g. c. m. of 360 and 112.

The process carried on may be written down as follows:—

112)360(3
336
24)112(4
96
16)24(1
16
8)16(2
16
0

and the rule for finding the greatest common measure of two numbers is,

I. Divide the greater of the two by the less.

II. Make the remainder a divisor, and the divisor a dividend, and find another remainder.

III. Proceed in this way until there is no remainder, and the last divisor is the greatest common measure required.

(83.) You may perhaps ask what will be the case when the two numbers have no common measure. The fact is, that there are, strictly speaking, no such numbers, because all numbers are measured by 1, that is, contain an exact number of units, and therefore 1 is a common measure of every two numbers. If they have no other common measure, the last divisor will be 1, as in the following example, where the greatest common measure of 87 and 25 is found.

```
25)87(3
   75
  ---
   12)25(2
      24
     ---
       1)12(12
         12
        ---
          0
```

EXERCISES.

What is the greatest common measure of 22464 and 17376, of 33197436 and 210131172, and of 1235540 and 196?

What are $36 \times 36 + 2 \times 36 \times 72 + 72 \times 72$
and $36 \times 36 \times 36 + 72 \times 72 \times 72$?
and what is their greatest common measure?

(84.) If two numbers be divisible by a third, and if the quotients be again divisible by a fourth, that third is not the greatest common measure. For example, 360 and 504 are both divisible by 4. The quotients are 90 and 126. Now 90 and 126 are both divisible by 9, the quotients of which division are 10 and 14. Now, by (73), dividing a number by 4, and then dividing the quotient by 9, is the same thing as dividing the number itself by 4 × 9, or by 36. Then, since 36 is a common measure of 360 and 504, and is greater than 4, 4 is not the greatest common measure. Again, since 10 and 14 are both divisible by 2, 36 is not the greatest common measure. It therefore follows, that, when two numbers are divided by their greatest common measure, the quotients have no common measure except 1 (83). Otherwise, the number which was called the greatest common measure in the last sentence is not so in reality.

(85.) When a first number contains a second, or is divisible by it without remainder, the first is called a multiple of the second. The words *multiple* and *measure* are thus connected: Since 4 is a measure of 24, 24 is a multiple of 4. The number 96 is a multiple of 8, 12, 24, 48, and several others. It is therefore called a *common multiple* of 8, 12, 24, 48, &c. The product of any two numbers is evidently a common multiple of both. Thus 36 × 8, or 288, is a common multiple of 36 and 8. But there are common multiples of 36 and 8 less than 288; and because it is convenient, when a common multiple of two quantities is wanted, to use the least of them, I now show how to find the least common multiple of two numbers.

(86.) Take, for example, 36 and 8. Find their greatest common measure, which is 4, and observe that 36 is 9 × 4, and 8 is 2 × 4. The quotients of 36 and 8, when divided by their greatest common measure, are therefore 9 and 2. Multiply these quotients together, and multiply the product by the greatest common measure, 4, which gives 9 × 2 × 4, or 72. This is a multiple of 8, or of 4 × 2, by (46); and also of 36, or of 4 × 9. It is also the least common multiple; but this cannot be proved to you, because the demonstration cannot be thoroughly understood without some knowledge of algebra. But you may satisfy yourself that it is the least in this case, and that the same process will give you the least common multiple in any other case which you may take. It is not even necessary that you should know it is the least. Whenever a common multiple is to be used, any one will do as well as the least. It is only to avoid large numbers that the least is used in preference to any other.

When the greatest common measure is 1, the least common multiple of the two numbers is their product.

The rule then is,—To find the least common multiple of two numbers, find their greatest common measure, and multiply one of the numbers by the quotient which the other gives when divided by the greatest common measure. To find the least common multiple of three numbers, find the least common multiple of the first two, and find the least common multiple of that multiple, and the third, and so on.

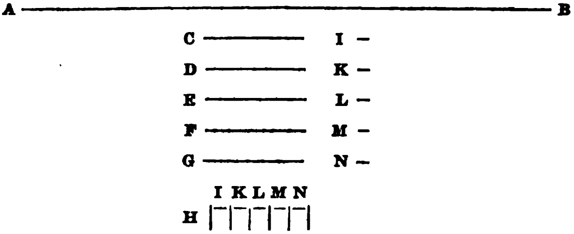
EXERCISES.

Find the least common multiple of 18, 54, and 136, of 22 and 99, of 33 and 144, and of 16, 46, and 1000.

SECTION V.

FRACTIONS.

(87.) Suppose it required to divide 49 yards into 5 equal parts, or, as it is called, to find the fifth part of 49 yards. If we divide 49 by 5, the quotient is 9, and the remainder is 4; that is (60), 49 is made up of 5 times 9 and 4. Let the line AB represent 49 yards:



and take 5 lines, C, D, E, F, and G, each 9 yards in length, and the line H, 4 yards in length. Then, since 49 is 5 nines and 4, C, D, E, F, G, and H are together equal to AB. Now divide H, which is 4 yards, into five equal parts, I, K, L, M, and N, and place one of these parts opposite to each of the lines C, D, E, F, and G. It follows that the ten lines, C, D, E, F, G, I, K, L, M, and N, are together equal to AB, or 49 yards. Now D and K together are of the same length as C and I together, and so are E and L, F and M, and G and N. Therefore C and I together, repeated 5 times, will be 49 yards; that is, C and I together make up the fifth part of 49 yards.

(88.) Now C is a certain number of yards, viz., 9; but I is a new sort of quantity, to which hitherto we have never come. It is not an exact number of yards, for it arises from dividing 4 yards into 5 parts, and taking one of those parts. It is the fifth part of 4 yards, and is called a FRACTION of a yard. It is written thus, $\frac{4}{5}$ (21), and is what we must add to 9 yards in order to make up the fifth part of 49 yards.

The same reasoning would apply to dividing 49 bushels of corn, or 49 acres of land, into 5 equal parts. We should find for the fifth part of the first 9 bushels, and the fifth part of 4 bushels; and for the second 9 acres, and the fifth part of 4 acres.

We say then, once for all, that the fifth part of 49 is 9 and $\frac{4}{5}$, or $9 + \frac{4}{5}$; which is usually written $9\frac{4}{5}$, or if we use signs, $\frac{49}{5} = 9\frac{4}{5}$.

EXERCISES.

What is the seventeenth part of 1237?

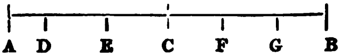
What are $\frac{10032}{1974}$, $\frac{663819}{23710}$, and $\frac{22773399}{2424}$?

(89.) By the term fraction is understood a part of any number, or what arises from dividing any number into a number of equal parts. Thus, $\frac{49}{5}$, $\frac{4}{5}$, $\frac{20}{7}$, are fractions. The term fraction even includes whole numbers*: for example, 17 is $\frac{17}{1}$, $\frac{34}{2}$, $\frac{51}{3}$, &c.

The upper line is called the *numerator*, the lower line is called the *denominator*, and both of these are called *terms* of the fraction.

(90.) By $\frac{2}{3}$ is meant the third part of 2. This is the same as twice the third part of 1.

To prove this, let AB be two yards, and divide each of the yards AC and CB into three equal parts.



Then, because AE, EF, and FB, are all equal to one another, AE is the third part of 2. It is therefore $\frac{2}{3}$. But AE is twice AD, and AD is the third part of one yard, or $\frac{1}{3}$; therefore $\frac{2}{3}$ is twice $\frac{1}{3}$; that is, in order to get the length $\frac{2}{3}$, it makes no difference whether we divide *two yards* at once into three parts, and take *one* of them, or whether we divide *one* yard into three parts, and take *two* of them. By the same reasoning $\frac{5}{8}$ may be found either by dividing 5 into 8 parts, and taking one of them, or by dividing 1 into 8 parts, and taking 5 of them.

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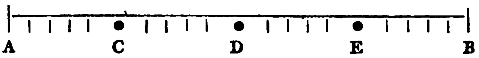
* Numbers which contain an exact number of units, such as 5, 7, 100, &c., are called *whole numbers, or integers,* when we wish to distinguish them from fractions.

In future, of these two meanings, I shall use that which is most convenient at the time, as it is proved that they are the same thing.

(91.) The value of a fraction is not altered by multiplying the numerator and denominator by the same quantity. Take the fraction $\frac{3}{4}$, multiply its numerator and denominator by 5, and it becomes $\frac{15}{20}$, which is the same thing as $\frac{3}{4}$; that is, one-twentieth part of 15 yards is the same thing as one-fourth of 3 yards: or, if our second meaning of the word fraction be used, you get the same length by dividing a yard into 20 parts and taking 15 of them, as you get by dividing it into 4 parts and taking 3 of them. To prove this,



let AB represent a yard: divide it into 4 equal parts, AC, CD, DE, and EB, and divide each of these parts into 5 equal parts. Then AE is $\frac{3}{4}$. But the second division cuts the line into 20 equal parts, of which AE contains 15. It is therefore $\frac{15}{20}$. Therefore $\frac{15}{20}$ and $\frac{3}{4}$ are the same thing. And since $\frac{3}{4}$ is made from $\frac{15}{20}$ by dividing both the numerator and denominator by 5, the value of a fraction is not altered by dividing both its numerator and denominator by the same quantity.

(92.) Though the two fractions $\frac{3}{4}$ and $\frac{15}{23}$ are the same in value, and either of them may be used for the other without error, yet the first is more convenient than the second, not only because you have a clearer idea of the fourth of three yards than of the twentieth part of fifteen yards, but because the numbers in the first, being smaller, are more convenient for multiplication and division. It is therefore useful, when a fraction is given, to find out whether its numerator and denominator have any common divisors or common measures. Now, in (82) was given a rule for finding the greatest common measure of any two numbers; and it was shown, that when the two numbers are divided by their greatest common measure, the quotients have no common measure except 1. Find the greatest common measure of the terms of the fraction, and divide them by that number. The fraction is then said to be *reduced to its lowest terms*, and is in the state in which it is most convenient to use it.

EXERCISES.

Reduce $\frac{17632}{9940}$, $\frac{22}{1650}$, and $\frac{3300}{22011110}$ to their lowest terms.

(93.) As we can by (91) multiply the numerator and denominator of a fraction by any number, without altering its value, we can now readily reduce two fractions to two others, which shall have the same value as the first two, and which shall have the same denominator. Take, for example, $\frac{2}{3}$ and $\frac{4}{7}$. Multiply both terms of $\frac{2}{3}$ by 7, and both terms of $\frac{4}{7}$ by 3. It then appears that

$$\frac{2}{3} \text{ is } \frac{2 \times 7}{3 \times 7} \text{ or } \frac{14}{21}$$

$$\frac{4}{7} \text{ is } \frac{4 \times 3}{7 \times 3} \text{ or } \frac{12}{21}$$

The two fractions are, therefore, the same as $\frac{14}{21}$ and $\frac{12}{21}$, which have the same denominator, and they are said to be *reduced to a common denominator.*

It is required to reduce $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$ to a common denominator. Multiply both terms of the first by the product of 6 and 9; of the second by the product of 10 and 9; and of the third by the product of 10 and 6. Then it appears (91) that

$$\frac{1}{10} \text{ is } \frac{1\times6\times9}{10\times6\times9} \text{ or } \frac{54}{540}$$

$$\frac{5}{6} \text{ is } \frac{5\times10\times9}{6\times10\times9} \text{ or } \frac{450}{540}$$

$$\frac{7}{9} \text{ is } \frac{7\times10\times6}{9\times10\times6} \text{ or } \frac{420}{540}$$

On looking at these last fractions, we see that all the numerators and the common denominator are divisible by 6, and (91) this division will not alter their values. On dividing the numerators and denominators of $\frac{54}{540}$, $\frac{450}{540}$, and $\frac{420}{540}$ by 6, the resulting fractions are, $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$. These are fractions with a common denominator, and which are the same as $\frac{1}{13}$, $\frac{5}{6}$, and $\frac{7}{9}$; and, therefore, these second fractions are a more simple answer to the question than the first. Observe also that 540 is one common multiple of 10, 6, and 9, namely, 10×6×9; but that 90 is *the least* common multiple of 10, 6, and 9 (86.). The following process, therefore, is shorter. To reduce $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$ to others having the same value and a common denominator, begin by finding the least common multiple of 10, 6, and 9, by the rule in (86), which is 90.

Observe that 10, 6, and 9 are contained in 90, 9, 15, and 10 times. Multiply both terms of the first by 9, of the second by 15, and of the third by 10, and the fractions thus produced are $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$, the same as before.

If one of the numbers be a whole number, it may be reduced to a fraction having the common denominator of the rest by (89).

EXERCISES.

Reduce $\frac{24}{29}$ and $\frac{1}{107}$ to a common denominator.

Reduce to a common denominator

$\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{14}$, $\frac{12}{21}$, and $\frac{3}{4}$.
$\frac{1}{100}$, $\frac{54}{10000}$, $\frac{632}{1000}$, and $\frac{1}{10}$.
$\frac{5}{10000}$, $\frac{7}{1000}$, $\frac{3}{100}$, and $\frac{2}{10}$.
7, 4, $\frac{22}{15}$, and $\frac{30}{7}$.

(94.) By reducing two fractions to a common denominator we are able to compare them, that is, to tell which is the greatest and which the least of the two. For example, take $\frac{1}{2}$ and $\frac{7}{15}$. These fractions reduced, without alteration of their value, to a common denominator, are $\frac{15}{30}$ and $\frac{14}{30}$. Of these the first must be the greatest, because (90) it may be obtained by dividing 1 into 30 equal parts, and taking 15 of them, whereas the second is made by taking 14 of those parts.

We can now also add and subtract the second fraction from the first. For the first fraction is made up of 15 of the 30 equal parts into which 1 is divided. The second fraction is 14 of those parts. The sum of the two, therefore, must be 15 + 14, or 29 of those parts; that is, $\frac{1}{2} + \frac{1}{15}$ is $\frac{29}{30}$. The difference of the two must be 15 − 14, or 1 of those parts; that is, $\frac{1}{2} - \frac{7}{15} = \frac{1}{30}$.

(95.) From the two last articles the following rules are obtained:

I. To compare, to add, or subtract fractions, first reduce them to a common denominator. When this has been done, of the fractions that is the greatest which has the greatest denominator.

Their sum has the sum of the numerators for its numerator, and the common denominator for its denominator.

Their difference has the difference of the numerators for its numerator, and the common denominator for its denominator.

EXERCISES.

What are $\frac{1}{100} + \frac{7}{22} + 9$, $\frac{6}{19} - \frac{4}{23}$, $\frac{2}{11} + \frac{1}{108} - \frac{1}{6}$, $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$, $\frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$, and $\frac{44}{1733} + \frac{62}{1408} - \frac{1}{7}$?

(96.) Suppose it required to add 6 and $\frac{4}{9}$. By (89) 6 is $\frac{54}{9}$, and $\frac{64}{9} + \frac{4}{9}$ is $\frac{68}{9}$; that is, $6 + \frac{4}{9}$, or, as it is usually written, $6\frac{4}{9}$, is $\frac{68}{9}$. The rule in this case is, Multiply the whole number by the denominator of the fraction, and to the product add the numerator of the fraction. The sum will be the numerator of the result, and the denominator of the fraction will be its denominator.

(97.) From the last rule it appears that $1723\frac{907}{10000}$ is $\frac{17230907}{10000}$, $667\frac{225}{1000}$ is $\frac{667225}{1000}$, and $23\frac{99}{1000000}$ is $\frac{23000099}{1000}$. Hence, when a whole number is to be added to a fraction whose denominator is 1 followed by *ciphers*, the number of which is not less than the number of *figures* in the numerator, the rule is, Write the whole number first, and then the numerator of the fraction, with as many ciphers between them as the number of *ciphers* in the denominator exceeds the number of *figures* in the numerator. This is the numerator of the result, and the denominator of the fraction is its denominator. If the number of ciphers in the denominator be equal to the number of figures in the numerator, write no ciphers between the whole number and the numerator.

EXERCISES.

Reduce the following mixed quantities to fractions: $1\frac{23707}{1000000}$, $2457\frac{6}{10}$, $1207\frac{299}{100000000}$, and $223\frac{2210}{10000}$.

(98.) Suppose it required to multiply $\frac{2}{3}$ by 4. This by (40) is taking $\frac{2}{3}$ four times and adding them together; that is, in finding $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$. This by (94) is $\frac{8}{3}$; so that to multiply a fraction by a whole number, the rule is, Multiply the numerator by the whole number, and let the denominator remain.

(99.) If the denominator of the fraction be divisible by the whole number, the rule may be stated thus: Divide the denominator of the fraction by the whole number, and let the numerator remain. For example, multiply $\frac{7}{36}$ by 6. This (98) is $\frac{42}{36}$, which, since the numerator and denominator are divisible by 6, is (91) the same as $\frac{7}{6}$. Now $\frac{7}{6}$ is made from $\frac{7}{36}$ in the manner stated in the rule.

(100.) Multiplication has been defined as the taking as many of one number as there are units in another. Thus, to multiply 12 by 7 is to take as many twelves as there are units in 7, or to take 12 as many times as you must take 1 in order to make 7. Thus, what is done with 1 in order to make 7, is done with 12 to make 7 times 12. For example,

7 is $1+1+1+1+1+1+1$,

7 times 12 is $12+12+12+12+12+12+12$.

When the same thing is done with two fractions, the result is still called their product, and the process is still called multiplication. There is this difference, that whereas a whole number is made by adding 1 to itself a number of times, a fraction is made by dividing 1 into a number of equal parts, and adding *one of these parts* to itself a number of times. For example, what is $\frac{3}{4}$ multiplied by $\frac{7}{8}$? Whatever is done with 1 in order to make $\frac{7}{8}$ must now be done with $\frac{3}{4}$; but to make $\frac{7}{8}$, 1 is divided into 8 parts, and 7 of them are taken. Therefore to make $\frac{3}{4} \times \frac{7}{8}$, $\frac{3}{4}$ must be divided into 8 parts, and 7 of them must be taken. Now $\frac{3}{4}$ is, by (91), the same thing as $\frac{24}{32}$. Since $\frac{24}{32}$ is made by dividing 1 into 32 parts and taking 24 of them, or, which is the same thing, taking 3 of them 8 times, if $\frac{24}{32}$ be divided into 8 equal parts, each of them is $\frac{3}{32}$; and if 7 of these parts be taken, the result is $\frac{21}{32}$ (98).

Therefore $\frac{3}{4}$ multiplied by $\frac{7}{8}$ is $\frac{21}{32}$; and the same reasoning may be applied to any other fractions. But $\frac{21}{32}$ is made from $\frac{3}{4}$ and $\frac{7}{8}$ by multiplying the two numerators together for the numerator, and the two denominators for the denominator, which furnishes a rule for the multiplication of fractions.

(101.) If this product $\frac{21}{32}$ is to be multiplied by a third fraction, for example by $\frac{5}{9}$, the result is, by the same rule, $\frac{135}{288}$, and so on. The general rule for multiplying any number of fractions together is therefore,

Multiply all the numerators together for the numerator, and all the denominators together for the denominator of the product.

(102.) Suppose it required to multiply together $\frac{15}{16}$ and $\frac{8}{10}$. The product may be written thus: $\frac{15 \times 8}{16 \times 10}$, and is $\frac{120}{160}$, which reduced to its lowest terms (92) is $\frac{3}{4}$. This result might have been obtained directly by observing that 15 and 10 are both measured by 5, and 8 and 16 are both measured by 8, and that the fraction may be written thus: $\frac{3 \times 5 \times 8}{2 \times 8 \times 2 \times 5}$. Divide both its numerator and denominator by 5×8, (91) and (73), and the result is at once $\frac{3}{4}$; therefore before proceeding to multiply any number of fractions together, if there be any numerator and any denominator which have a common measure, divide them both by that common measure, and use the quotients instead of the dividends.

A whole number may be considered as a fraction whose denominator is 1 (thus 16 is $\frac{16}{1}$) (89); and the same rule will apply when one or more of the quantities are whole numbers.

EXERCISES.

Multiply together $\frac{1}{2}$, 3, $\frac{2}{3}$, 4 and $\frac{7}{10}$,
. . . . $\frac{22}{36}$ $\frac{21}{11}$ and $\frac{44}{97}$.

From 100 acres of ground, two-thirds of them are taken away; 50 acres are then added to the result, and $\frac{5}{7}$ of the whole is taken. What number of acres does this produce?

A can reap a field of corn by himself in 6 days, B can do it in $\frac{4}{7}$ of that time, and C in $\frac{3}{11}$ of that time. How much can all do together in half a-day?

(103.) In dividing one whole number by another, for example 108 by 9, this question is asked:—Can we by the addition of any number of nines produce 108?—and if so, how many nines will be sufficient for that purpose?

Suppose we take two fractions, for example $\frac{4}{5}$ and $\frac{2}{3}$, and ask, Can we by dividing $\frac{4}{5}$ into some number of equal parts, and adding a number of these parts together, produce $\frac{2}{3}$?—if so, into *how many parts* must we divide $\frac{4}{5}$, and *how many of them* must we add together? The solution of this question is still called the division of $\frac{2}{3}$ by $\frac{4}{5}$; and the fraction whose denominator is the number of parts into which $\frac{4}{5}$ is divided, and whose numerator is the number of them which is taken, is called the quotient. The solution of this question is as follows:

Reduce both these fractions to a common denominator (93), which does not alter their value (91.): they then become $\frac{10}{15}$ and $\frac{12}{15}$. The question now is, to divide $\frac{12}{15}$ into a number of parts, and to produce $\frac{10}{15}$ by taking a number of these parts. Since $\frac{12}{15}$ is made by dividing 1 into 15 parts and taking 12 of them, if we divide $\frac{12}{15}$ into 12 equal parts, each of these parts is $\frac{1}{15}$: if we take 10 of these parts, the result is $\frac{10}{15}$. Therefore, in order to produce $\frac{10}{15}$ or $\frac{2}{3}$ (91), we must divide $\frac{12}{15}$ or $\frac{4}{5}$ into 12 parts, and take 10 of them; that is, the quotient is $\frac{10}{12}$. If we call $\frac{2}{3}$ the dividend and $\frac{4}{5}$ the divisor, as before, the quotient in this case is derived from the following rule, which the same reasoning will show to apply to other cases:

The numerator of the quotient is the numerator of the dividend multiplied by the denominator of the divisor. The denominator of the quotient is the denominator of the dividend multiplied by the numerator of the divisor.

(104.) By taking the following instance, we shall see that this rule can be sometimes simplified. Divide $\frac{16}{33}$ by $\frac{28}{15}$. Observe that 16 is 4×4, and 28 is 4×7: 33 is 3×11, and 15 is 3×5; therefore the two fractions are $\frac{4 \times 4}{3 \times 11}$ and $\frac{4 \times 7}{3 \times 5}$, and their quotient, according to the rule, is $\frac{4 \times 4 \times 3 \times 5}{3 \times 11 \times 4 \times 7}$, in which 4×3 is found both in the numerator and denominator. The fraction is therefore (91) the same as $\frac{4 \times 5}{11 \times 7}$, or $\frac{20}{77}$. The rule of the last article therefore admits of this addition:—If the two numerators or the two denominators have a common measure, divide by that common measure, and use the quotients instead of the dividends.

(105.) In dividing a fraction by a whole number, for example $\frac{2}{3}$ by 15, consider 15 as the fraction $\frac{15}{1}$. The rule gives $\frac{2}{45}$ as the quotient. Therefore, to divide a fraction by a whole number, multiply the denominator by that whole number.

EXERCISES.

Divide $\frac{44}{5}$ by $\frac{108}{10}$, 163 by $3\frac{1}{4}$, and $\frac{25}{9}$ by 160.

What are $\dfrac{\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} - \frac{2}{17} \times \frac{2}{17} \times \frac{2}{17}}{\frac{1}{5} - \frac{1}{7}}$,

$\dfrac{\frac{8}{11} \times \frac{8}{11} - \frac{3}{11} \times \frac{3}{11}}{\frac{8}{11} - \frac{3}{11}}$, and $\dfrac{\frac{26}{53}}{\frac{19}{44}}$?

What part is $\frac{104}{91}$ of $\frac{92}{13}$, $462\frac{2}{17}$ of 1803, and 1688 of 1830?

A can reap a field in 12 days, B in 6, and C in 4 days. In what time can they all do it together?

SECTION VI.

DECIMAL FRACTIONS.

(106.) We have seen (94) (103) the necessity of reducing fractions to a common denominator, in order to compare their magnitudes. We have seen also how much more readily operations are performed upon fractions which have the same, than upon those which have different, denominators. On this account it has long been customary, in all those parts of mathematics where fractions are often required, to use none but such as either have, or can be very easily reduced to others having, the same denominator. Now, of all numbers, those which can be most easily managed are such as 10, 100, 1000, &c., where 1 is followed by ciphers. These are called *decimal numbers*, and a fraction, whose denominator is any one of them, is called a DECIMAL FRACTION, or, more commonly, a DECIMAL.

(107.) A whole number may be reduced to a decimal fraction, or one decimal fraction to another, with the greatest ease. For example, 94 is $\frac{940}{10}$, or $\frac{9400}{100}$, or $\frac{94000}{1000}$ (89); $\frac{3}{10}$ is $\frac{30}{100}$, or $\frac{300}{1000}$, or $\frac{3000}{10000}$ (91). The addition of a cipher to the right-hand of any number is the same thing as multiplying that number by 10 (47), and this may be done as often as we please in the numerator of a fraction, provided it is done as often in the denominator (91).

(108.) The next question is, is it possible to reduce a fraction which is not a decimal to another which is, without altering its value? Take, for example, the fraction $\frac{7}{16}$, multiply both the numerator and denominator successively by 10, 100, 1000, &c., which will give a series of fractions, each of which is equal to $\frac{7}{16}$ (91), viz. $\frac{70}{160}$, $\frac{700}{1600}$, $\frac{7000}{16000}$, $\frac{70000}{160000}$, &c. The denominator of each of these fractions can be divided without remainder by 16, the quotients of which divisions form the series, of decimal numbers 10, 100, 1000, 10000, &c. If therefore one of the numerators is divisible by 16, the fraction to which that numerator belongs has a numerator and denominator both divisible by 16. When that division has been made which (91) does not alter the value of the fraction, we shall have a fraction whose denominator is one of the series 10, 100, 1000, &c., and which is equal in value to $\frac{7}{16}$. The question is then reduced to finding the first of the numbers 70, 700, 7000, 70000, &c., which can be divided by 16 without remainder.

Divide these numbers, one after the other, by 16, as follows:

16)70(4	16)700(43	16)7000(437	16)70000(4375
64	64	64	64
6	60	60	60
	48	48	48
	12	120	120
		112	112
		8	80
			80
			0

It appears, then, that 70000 is the first of the numerators which is divisible by 16. But it is not necessary to write down each of these divisions, since it is plain that the last contains all which came before. It will do then to proceed at once, as if the number of ciphers were without end, to stop when the remainder is nothing, and then count the number of ciphers which have been used. In this case, since 70000 is 16 × 4375, $\frac{70000}{160000}$, which is $\frac{16\times4375}{16\times10000}$, or $\frac{4375}{10000}$, is the fraction required.

Therefore, to reduce a fraction to a decimal fraction, annex ciphers to the numerator, and divide by the denominator until there is no remainder. The quotient will be the numerator of the required fraction, and the denominator will be unity, followed by as many ciphers as were used in obtaining the quotient.

EXERCISES.

Reduce to decimal fractions

$\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{25}$, $\frac{1}{50}$, $\frac{3927}{1250}$, and $\frac{453}{625}$.

(109.) It will happen in most cases that the addition of ciphers to the numerator will never make it divisible by the denominator without remainder. For example, try to reduce $\frac{1}{7}$ to a decimal fraction.

$$\begin{array}{r} 7)1000000000000000000\text{, \&c.} \\ \hline 142857142857142857\text{, \&c.} \end{array}$$

The quotient here is a continual repetition of the figures 1, 4, 2, 8, 5, 7, in the same order. Therefore $\frac{1}{7}$ cannot be reduced to a decimal fraction. But nevertheless, if we take as a numerator any number of figures from the quotient 142857142857, &c., and as a denominator, 1 followed by as many ciphers as were used in making that part of the quotient, we shall get a fraction which differs very little from $\frac{1}{7}$, and which will differ still less from it if we put more figures in the numerator, and more ciphers in the denominator.

Thus, $\frac{1}{10}$ is less than $\frac{1}{7}$ by $\frac{3}{70}$ which is not so much as $\frac{1}{10}$
$\frac{14}{100}$ .... $\frac{1}{7}$ .... $\frac{2}{700}$ .... $\frac{1}{100}$
$\frac{142}{1000}$ .... $\frac{1}{7}$ .... $\frac{6}{7000}$ .... $\frac{1}{1000}$
$\frac{1428}{10000}$ .... $\frac{1}{7}$ ... $\frac{4}{70000}$ .... $\frac{1}{10000}$
$\frac{14285}{100000}$ .... $\frac{1}{7}$ .. $\frac{5}{700000}$ .... $\frac{1}{100000}$
$\frac{142857}{1000000}$ .... $\frac{1}{7}$ .. $\frac{1}{7000000}$ .... which is .... $\frac{1}{1000000}$
&c. &c. &c. &c.

EXERCISES.

Make a similar table with these fractions, $\frac{17}{47}$, $\frac{201}{19}$, $\frac{3}{26}$, $\frac{4}{55}$, and $\frac{10817}{2377}$.

In the first column is a series of decimal fractions, which come nearer and nearer to $\frac{1}{7}$, as the third column shows. Therefore, though we cannot find a decimal fraction which is exactly $\frac{1}{7}$, we can find one which differs from it as little as we please.

(110.) You will here naturally ask, What is the use of decimal fractions, if the greater number of fractions cannot be reduced at all to decimals? The answer is this: The addition, subtraction, multiplication, and division of decimal fractions, are much easier than those of common fractions; and though we cannot reduce all common fractions to decimals, yet we can find decimal fractions so near to each of them, that the error arising from using the decimal instead of the common fraction will not be perceptible. For example, if we suppose an inch to be divided into ten million of equal parts, one of those parts by itself will not be visible to the eye. Therefore, in finding a length, an error of a ten-millionth part of an inch is of no consequence, even where the finest measurement is necessary. Now, by carrying on the table in (109), we shall see that $\frac{1428571}{10000000}$ does not differ from $\frac{1}{7}$ by $\frac{1}{10000000}$; and if these fractions represented parts of an inch, the first might be used for the second, since the difference is not perceptible.

EXERCISES.

Find decimal fractions which do not differ from the following fractions by $\frac{1}{100000000}$.

$\frac{1}{3}$, $\frac{4}{7}$, $\frac{22}{53}$, $\frac{113}{355}$, and $\frac{355}{113}$.

(111.) Every decimal may be immediately reduced to a quantity, either consisting of a whole number and more simple decimals, or of more simple decimals alone, having one figure only in each of the numerators. Take, for example, $\frac{147326}{1000}$. By (97) $\frac{147326}{1000}$ is $147\frac{326}{1000}$; and since 326 is made up of 300, and 20, and 6; by (94) $\frac{326}{1000} = \frac{300}{1000} + \frac{20}{1000} + \frac{6}{1000}$. But (91) $\frac{300}{1000}$ is $\frac{3}{10}$, and $\frac{20}{1000}$ is $\frac{2}{1000}$. Therefore $\frac{147326}{1000}$ is made up of $147 + \frac{3}{10} + \frac{2}{100} + \frac{6}{1000}$. Now take any number, for example, 147326, and form a number of fractions, having for their numerators this number, and for their denominators 1, 10, 100, 1000, 10000, &c., and reduce these fractions into numbers and more simple decimals, in the foregoing manner, which will give the table on the opposite page.

EXERCISES.

Reduce the following fractions into a series of numbers and more simple fractions:

$\frac{31415926}{10}$, $\frac{31415926}{100}$, &c.

$\frac{2700031}{10}$, $\frac{2700031}{100}$, &c.

$\frac{2073000}{10}$, $\frac{2073000}{100}$, &c.

$\frac{3331303}{1000}$, $\frac{3331303}{10000}$, &c.

TABLE OF DECOMPOSITION OF DECIMAL FRACTIONS.

$$\frac{147326}{1} = 147326$$

$$\frac{147326}{10} = 14732 + \frac{6}{10}$$

$$\frac{147326}{100} = 1473 + \frac{2}{10} + \frac{6}{100}$$

$$\frac{147326}{1000} = ..\ 147 + \frac{3}{10} + \frac{2}{100} + \frac{6}{1000}$$

$$\frac{147326}{10000} = ...\ 14 + \frac{7}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{6}{10000}$$

$$\frac{147326}{100000} = ...\ 1 + \frac{4}{10} + \frac{7}{100} + \frac{3}{1000} + \frac{2}{10000} + \frac{6}{100000}$$

$$\frac{147326}{1000000} = ...\ \frac{1}{10} + \frac{4}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{2}{100000} + \frac{6}{1000000}$$

$$\frac{147326}{10000000} = ...\ \frac{1}{100} + \frac{4}{1000} + \frac{7}{10000} + \frac{3}{100000} + \frac{2}{1000000} + \frac{6}{10000000}$$

N.B. The student should write this table himself, and then proceed to make similar tables from the exercises in (111).

(112.) If, in this table, and others made in the same manner, you look at those fractions which contain a whole number, you will see that they may be made thus: Mark off, from the right hand of the numerator, as many *figures* as there are *ciphers* in the denominator, by a point, or any other convenient mark.

This will give 14732·6 when the fraction is $\frac{147326}{10}$
. . . . 1473·26 . . . . $\frac{147326}{100}$
. . . . 147·326 . . . . $\frac{147326}{1000}$
&c. &c.

The figures on the left of the point, by themselves make the whole number which the fraction contains. Of those on its right, the first is the numerator of the fraction whose denominator is 10, the second of that whose denominator is 100, and so on. We now come to those fractions which do not contain a whole number.

(113.) The first of these is $\frac{147326}{1000000}$, in which the number of ciphers in the denominator is the same as the number of figures in the numerator. If we still follow the same rule, and mark off all the figures, by placing the point before them all, thus, ·147326, the rule still holds good; for, on looking at $\frac{147326}{1000000}$ in the table, we find it is

$$\frac{1}{10}+\frac{4}{100}+\frac{7}{1000}+\frac{3}{10000}+\frac{2}{100000}+\frac{6}{1000000}$$

The next fraction is $\frac{147326}{10000000}$, which we find by the table to be

$$\frac{1}{100}+\frac{4}{1000}+\frac{7}{10000}+\frac{3}{100000}+\frac{2}{1000000}+\frac{6}{10000000}$$

In this 1 is not divided by 10 but by 100; if, therefore, we put a point before the whole, the rule is not true, for the first figure in the left of the point has the denominator which, according to the rule, the second ought to have, the second that which the third ought to have, and so on. In order to keep the same rule for this case, we must contrive to make 1 the second figure on the right of the point instead of the first. This may be done by placing a cipher between it and the point, thus, .0147326. Here the rule holds good, for by that rule this fraction is

$$\frac{0}{10}+\frac{1}{100}+\frac{4}{1000}+\frac{7}{10000}+\frac{3}{100000}+\frac{2}{1000000}+\frac{6}{10000000},$$

which is the same as the foregoing value, since $\frac{0}{10}$ is 0, and need not be reckoned.

Similarly, when there are two ciphers more in the denominator than there are figures in the numerator, the rule will be true if we place two ciphers between the point and the numerator. The rule, therefore, stated fully is this:—

To reduce a decimal fraction to a number and more simple decimals, or to more simple decimals alone, if it does not contain a whole number, mark off by a point as many figures from the numerator as there are ciphers in the denominator. If the numerator has not places enough for this, write as many ciphers before it as it wants places, and place the point before these ciphers. Then, if there are any figures before the point, they make the *whole number* which the fraction contains. The first figure after the point with the denominator 10, the second with the denominator 100, and so on, are the *fractions* of which the first fraction is composed.

(114.) Decimal fractions are not usually written at full length. It is more convenient to write the numerator only, and to cut off from the numerator as many figures as there are ciphers in the denominator, when that is possible, by a point. When there are more ciphers in the denominator than figures in the numerator, as many ciphers are placed before the numerator as will supply the deficiency, and the point is placed before the ciphers. Thus .7 will be used in future to denote $\frac{7}{10}$, .07 for $\frac{7}{100}$, and so on. The following tables will give the whole of this notation at one view, and will show its connexion with the decimal notation explained in the first section. You will observe that the numbers on the right of the decimal point stand for units *divided* by 10, 100, 1000, &c. while those on the left, with the exception of the units themselves, are units multiplied by 10, 100, 1000, &c.

I. 123.4 stands for $\frac{1234}{10}$ or $123\frac{4}{10}$ or $123+\frac{4}{10}$

12.34 .... $\frac{1234}{100}$ or $12\frac{34}{100}$ or $12+\frac{3}{10}+\frac{4}{100}$

1.234 .... $\frac{1234}{1000}$ or $1\frac{234}{1000}$ or $1+\frac{2}{10}+\frac{3}{100}+\frac{4}{1000}$

.1234 ..... $\frac{1234}{10000}$ .... or .... $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{4}{10000}$

.01234 .... $\frac{1234}{100000}$ .... or .... $\frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}+\frac{4}{100000}$

.001234 .. $\frac{1234}{1000000}$ .... or .... $\frac{1}{1000}+\frac{2}{10000}+\frac{3}{100000}+\frac{4}{1000000}$

II. .1003 is $\frac{1003}{10000}$ or $\frac{1}{10}+\frac{3}{10000}$

10.03 is $\frac{1003}{100}$ or $10+\frac{3}{100}$

100.3 is $\frac{1003}{10}$ or $100+\frac{3}{10}$

III. In 1234.56789 inches the 1 is 1000 inches

2 is 200 . . .

3 is 30 . . .

4 is 4 . . .

5 is $\frac{5}{10}$ of an inch

6 is $\frac{6}{100}$ . . .

7 is $\frac{7}{1000}$ . . .

8 is $\frac{8}{10000}$ . . .

9 is $\frac{9}{100000}$ . . .

(115.) The ciphers on the right hand of the decimal point serve the same purpose as the ciphers in (12). They are not counted as any thing themselves, but serve to show the place in which the accompanying numbers stand. They might be dispensed with by writing the numbers in ruled columns, as in the first section. They are distinguished from the numbers which accompany them by calling the latter *significant figures.* Thus .0003747 is a decimal of seven places with four significant figures, .346 is a decimal of three places with three significant figures, &c.

(116.) The value of a decimal is not altered by putting any number of ciphers on its right. Take, for example, .3 and .300. The first (114) is $\frac{3}{10}$, and the second $\frac{300}{1000}$; the second of which is made from the first by multiplying both its numerator and denominator by 100, and (91) is the same quantity.

(117.) To reduce two decimals to a common denominator, put as many ciphers on the right of that which has the smallest number of places as will make the number of places in both fractions the same. Take, for example, ·54 and 4·3297. The first is $\frac{54}{100}$ and the second $\frac{43297}{10000}$. Multiply the numerator and denominator of the first by 100, (91) which reduces it to $\frac{5400}{10000}$, which has the same denominator as $\frac{43297}{10000}$. But $\frac{5400}{10000}$ is ·5400 (114). In whole numbers the decimal point should be placed at the end: thus 129 should be written 129·. It is, however, usual to omit the point, but you must recollect that 129 and 129·000 are the same thing, since the first is 129 and the second $\frac{129000}{1000}$.

(118.) The rules which were given in the last chapter for addition, subtraction, multiplication, and division apply to all fractions, and therefore to decimal fractions among the rest. But the way of writing decimal fractions, which is explained in this chapter, makes the application of these rules more simple. We proceed to the different cases.

Suppose it required to add 42·634, 45·2806, 2·001, and 54. By (94) these must be reduced to a common denominator, which is done (117) by writing them as follows: 42·6340, 45·2806, 2·0010, and 54·0000. These are decimal fractions whose numerators are 426340, 452806, 20010 and 540000, and whose common denominator is 10000. By (94) their sum is $\frac{426340+452806+20010+540000}{10000}$, which is $\frac{1439156}{10000}$ or 143·9156. The simplest way of doing this is as follows: write the decimals down under one another, so that the decimal points may fall under one another, thus:

$$\begin{array}{r@{\cdot}l} 42 & 634 \\ 45 & 2806 \\ 2 & 001 \\ 54 & \\ \hline 143 & 9156 \end{array}$$

Add the different columns together as in common addition, and place the decimal point under the other decimal points.

EXERCISES.

What is 1527 + 64·732094 + 2·0013 + ·00001974,
2276·3 + ·107 + ·9 + 26·3172 + 56732·001
and 1·11 + 7·7 + ·0039 + ·00142 + ·8838 ?

(119.) Suppose it required to subtract 91·07324 from 137·321. These fractions when reduced to a common denominator are 91·07324 and 137·32100 (117). Their difference is therefore $\frac{13732100-9107324}{100000}$, which is $\frac{4624776}{100000}$ or 46·24776. This may be most simply done as follows: write the less number under the greater, so that its decimal point may fall under that of the greater, thus:

$$\begin{array}{r@{\cdot}l} 137 & 321 \\ 91 & 07324 \\ \hline 46 & 24776 \end{array}$$

Subtract the lower from the upper line, and wherever there is a figure in one line and not in the other, proceed as if there were a cipher in the vacant place.

EXERCISES.

What is 12362 − 274·22107 + ·5
9976·2073942 − ·00143976728
and 1·2 + ·03 + ·004 − ·0005 ?

(120.) The multiplication of a decimal by 10, 100, 1000, &c. is performed by merely moving the decimal point to the right. Suppose, for example, 13·2079 is to be multiplied by 100. The decimal is $\frac{132079}{10000}$, which multiplied by 100 is (99) $\frac{132079}{100}$ or 1320·79. From this and other instances we get the following rule:—To multiply a decimal fraction by a decimal number (106), move the decimal point as many places to the right as there are ciphers in the decimal number. When this cannot be done, add ciphers to the right of the decimal (116) until it can.

(121.) Suppose it required to multiply 17·036 by 4·27. The first of these decimals is $\frac{17036}{1000}$, and the second $\frac{427}{100}$. By (100) the product of these fractions has for its numerator the product of 17036 and 427, and for its denominator the product of 1000 and 100; therefore this product is $\frac{7274372}{100000}$ or 72·74372. This may be done more shortly by multiplying the two numbers 17036 and 427, and cutting off by the decimal point as many places as there are decimal places both in 17·036 and 4·27.

(122.) This question now arises:—What if there should not be as many figures in the product as there are decimal places in the multiplier and multiplicand? To see what must be done in this case, multiply ·172 by ·101 or $\frac{172}{1000}$ and $\frac{101}{1000}$. The product of these two is $\frac{17372}{1000000}$ or ·017372 (121). Therefore, when the number of places in the product is not sufficient to allow the rule of the last article to be followed, as many ciphers must be placed at the beginning as will make up the deficiency.

ADDITIONAL EXAMPLES.

$\cdot 001 \times \cdot 01$ is $\cdot 0001$
$56 \times \cdot 0001$ is $\cdot 0056$.

EXERCISES.

Show that

$3{\cdot}002 \times 3{\cdot}002 = 3 \times 3 + 2 \times 3 \times {\cdot}002 + {\cdot}002 \times {\cdot}002$

$11{\cdot}5609 \times 5{\cdot}3191 = 8{\cdot}44 \times 8{\cdot}44 - 3{\cdot}1209 \times 3{\cdot}1209$

$8{\cdot}217 \times 10{\cdot}001 = 8 \times 10 + 8 \times {\cdot}001 + 10 \times {\cdot}217 + {\cdot}001 \times {\cdot}217$.

(123.) The division of a decimal by a decimal number, such as 10, 100, 1000, &c. is performed by merely moving the decimal point as many places to the left as there are ciphers in the decimal number. If there are not enough places in the dividend to allow of this, annex ciphers to the beginning of it until there are. For example, divide 1734·229 by 1000: the decimal fraction is $\frac{1734229}{1000}$, which divided by 1000 (105) is $\frac{1734229}{1000000}$ or 1·734229. If in the same way 1·2106 be divided by 10000, the result is ·00012106.

(124.) Before proceeding to shorten the rule for the division of one decimal fraction by another, it will be necessary to resume what was said in (108) upon the reduction of any fraction to a decimal fraction. It was there shown that $\frac{7}{16}$ is the same fraction as $\frac{4375}{10000}$ or ·4375. As another example, convert $\frac{3}{128}$ into a decimal fraction. Follow the same process as in (108) thus:

```
128)300000000000(234375
     256
    -----
      440
      384
     -----
       560
       512
      -----
        480
        384
       -----
         960
         896
        -----
          640
          640
         -----
            0
```

Since 7 ciphers are used, it appears that 30000000 is the first of the series 30, 300, &c. which is divisible by 128; and therefore $\frac{3}{128}$, or, which is the same thing (91), $\frac{30000000}{1280000000}$ is equal to $\frac{234375}{10000000}$ or ·0234375 (114).

From these examples the rule for reducing a fraction to a decimal appears to be, Annex ciphers to the numerator sufficient to make a number greater than the denominator; divide by the numerator and annex a cipher to each remainder, proceeding exactly as if the numerator had an unlimited number of ciphers annexed to it, and were to be divided by the denominator. Continue this process until there is no remainder, and observe how many ciphers have been used. Place the decimal point in the quotient so as to cut off as many figures as you have used ciphers; and if there be not figures enough for this, annex ciphers to the beginning until there are.

(125.) From what was shown in (109) it appears that it is not every fraction which can be reduced to a decimal fraction. It was there shown, however, that there is no fraction to which we may not find a decimal fraction as near as we please. Thus, $\frac{1}{10}$, $\frac{14}{100}$, $\frac{142}{1000}$, $\frac{1428}{10000}$, $\frac{14285}{100000}$, &c. or ·1, ·14, ·142, ·1428, ·14285, were shown to be fractions which approach nearer and nearer to $\frac{1}{7}$. To find either of these fractions, the rule is the same as that in the last article, with this exception, that, I., instead of stopping when there is no remainder, which never happens, stop at any part of the process, and make as many decimal places in the quotient as are equal in number to the number of ciphers which have been used, annexing ciphers to the beginning when this cannot be done, as before. II., instead of obtaining a fraction which is exactly equal to the fraction from which we set out, we get a fraction which is very near to it, and may get one still nearer by using more of the quotient. Thus, ·1428 is very near to $\frac{1}{7}$, but not so near as ·142857; nor is this last, in its turn, so near as ·142857142857, &c.

(126.) If there should be ciphers in the numerator of a fraction, these must not be reckoned with the number of ciphers which are necessary in order to follow the rule for changing it into a decimal fraction. Take, for example, $\frac{100}{125}$. Annex ciphers to the numerator, and divide by the denominator. It appears that 1000 is divisible by 125, and that the quotient is 8. One cipher only has been annexed to the numerator, and therefore 100 divided by 125, is ·8. Had the fraction been $\frac{1}{125}$, since 1000 divided by 25 gives 8, and three ciphers have been annexed to the numerator, the fraction is ·008.

(127.) Suppose that the given fraction has ciphers at the right of its denominator; for example, $\frac{31}{2500}$. The annexing a cipher to the numerator is the same thing as taking one away from the denominator; for (91) $\frac{310}{2500}$ is the same thing as $\frac{31}{250}$, and $\frac{3100}{2500}$, as $\frac{31}{25}$. The rule therefore is, in this case, Take away the ciphers from the denominator; annex ciphers to the numerator; proceed as before; and in counting how many ciphers have been used, reckon not only the ciphers which have been annexed to the numerator, but also those which have been taken away from the denominator.

EXERCISES.

Reduce the following fractions to decimal fractions:

$$\frac{1}{800}, \frac{36}{1250}, \frac{297}{64}, \text{ and } \frac{1}{128}.$$

Find decimals of 6 places very near to the following fractions:

$$\frac{27}{49}, \frac{156}{33}, \frac{22}{37000}, \frac{194}{13}, \text{ and } \frac{2637}{9907}.$$

(128.) From (104) it appears, that if two fractions have the same denominator, the first may be divided by the second by dividing the numerator of the first by the numerator of the second. Now suppose it is required to divide 17·762 by 6·25. These fractions (117), when reduced to a common denominator, are 17·762 and 6·250, or $\frac{17762}{1000}$ and $\frac{6250}{1000}$. Their quotient is therefore $\frac{17762}{6250}$, which must now be reduced to a decimal fraction by the last rule. The process at full length is as follows: leave out the cipher in the denominator, and annex ciphers to the numerator, or, which will do as well, to the remainders, when it becomes necessary, and divide as in (124).

```
625)17762(284192
    1250
    ----
     5262
     5000
     ----
      2620
      2500
      ----
       1200
        625
       ----
        5750
        5625
        ----
         1250
         1250
         ----
            0
```

Here four ciphers have been annexed to the numerator, and one has been taken from the denominator. Make five decimal places in the quotient, when there becomes 2·84192, and this is the quotient of 17·762 divided by 6·25.

(129.) The rule for division of one decimal by another is as follows. Throw aside the decimal points, annex ciphers to the numerator, and proceed as in ordinary division. When the remainder becomes nothing, count the number of ciphers which have been used, and mark off as many places from the right of the quotient, using ciphers if necessary; then observe how many more decimal places there are in the dividend than are in the divisor, or in the divisor than are in the dividend, and carry the decimal point as many places to the left in the first case, and to the right in the second, supplying any deficiency of places in either case by ciphers.

EXAMPLES.

$$\frac{3{\cdot}1}{{\cdot}0025} = 1240 \qquad \frac{{\cdot}00062}{{\cdot}64} = {\cdot}00096875.$$

If the remainder never becomes 0, the same rule will apply, if we stop at any part of the quotient, and count the number of ciphers, and proceed afterwards in the same manner.

EXERCISES.

Show that $\frac{15{\cdot}006 \times 15{\cdot}006 - {\cdot}004 \times {\cdot}004}{15{\cdot}01} = 15{\cdot}002$, and that $\frac{{\cdot}01 \times {\cdot}01 \times {\cdot}01 + 2{\cdot}9 \times 2{\cdot}9 \times 2{\cdot}9}{2{\cdot}91} = 2{\cdot}9 \times 2{\cdot}9$ $- 2{\cdot}9 \times {\cdot}01 + {\cdot}01 \times {\cdot}01$.

What are $\frac{1}{3{\cdot}14159}$, $\frac{1}{2{\cdot}7182818}$, and $\frac{365}{{\cdot}18349}$, as far as 6 places of decimals.

Calculate 10 terms of each of the following series as far as 5 places of decimals:

$$1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \&c.$$

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \&c.$$

$$\tfrac{80}{81} + \tfrac{81}{82} + \tfrac{82}{83} + \tfrac{83}{84} + \tfrac{84}{85} + \&c.$$

BOOK II.

COMMERCIAL ARITHMETIC.

SECTION I.

WEIGHTS, MEASURES, &C.

(130.) In making the calculations which are necessary for the business of commerce, no more processes are required than those which have been explained in the preceding book. But there is still one thing wanted—not to ensure the accuracy of our calculations, but to enable us to compare and judge of their results. We have hitherto made use of a single unit (13), and have treated of other quantities, which are made up of a number of units, in Sections 2, 3, and 4, and of those which contain parts of that unit in Sections 5 and 6. Thus, if we are talking of distances, and take a mile as the unit, any other length is either a certain number of miles, or a certain number of parts of a mile, and (1 meaning one mile) may be expressed either by a whole number or a fraction. But we can easily see that in many cases inconveniences would arise.

Suppose, for example, I say, that the length of one room is $\frac{1}{180}$ of a mile, and of another $\frac{1}{174}$, of a mile, what idea can we form as to how much the second is longer than the first? It is necessary to have some smaller measure, and if we divide a mile into 1760 equal parts, and call each of these parts a yard, we shall find that the length of the first room is 9 yards and $\frac{7}{9}$ of a yard, and that of the second 10 yards and $\frac{10}{87}$ of a yard. From this we form a much better notion of these different lengths, but still not very perfect, on account of the fractions $\frac{7}{9}$ and $\frac{10}{87}$. To get a clearer idea of these, suppose the yard to be divided into 3 equal parts, and each of these parts to be called a foot; then $\frac{7}{9}$ of a yard contains $2\frac{1}{3}$ feet, and $\frac{10}{87}$ of a yard contains $\frac{30}{87}$ of a foot, or a little more than $\frac{1}{3}$ of a foot. Therefore the length of the first room is now 9 yards, 2 feet, and $\frac{1}{3}$ of a foot; that of the second is 10 yards and a little more than $\frac{1}{3}$ of a foot. We see then the convenience of having large measures for large quantities, and smaller measures for small ones; but this is done for convenience only, for it is *possible* to perform calculations upon any sort of quantity, with one measure alone, as certainly as with more than one.

The measures which are used in this country are not those which would have been chosen had they been made all at one time, and by a people well acquainted with arithmetic and natural philosophy. Of these the most important are those which are contained in the following tables: the student should commit to memory those parts which are marked in italics. The abbreviations which are made use of are marked in the last column.

(131.) MEASURES OF MONEY.

The lowest coin is a farthing, which is marked thus, ¼, being one fourth of a penny.

2 *Farthings*	are	1 *halfpenny*	½*d.*
2 *Halfpence*		1 *penny*	1*d.*
12 *Pence*		1 *shilling*	1*s.*
20 *Shillings*		1 *pound* * or *sovereign*	£1.
21 *Shillings*		1 *guinea* †.	

(132.) MEASURES OF LENGTH.

The lowest measure is a barleycorn.

3 *Barleycorns*	are	1 *inch*	1 in.
12 *Inches*		1 *foot*	1 ft.
3 *Feet*		1 *yard*	1 yd.
5½ *Yards*		1 *pole*	1 po.
40 *Poles* or 220 *yards*		1 *furlong*	1 fur.
8 *Furlongs* or 1760 *yards*		1 *mile*	1 mi.
3 *Miles*		1 *league*	1 lea.

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* The English pound is generally called a *pound sterling*, which distinguishes it from the weight called a pound.

† The coin called a guinea is now no longer in use, but the name is still given from custom to 21 shillings. The pound, which was not a coin, but a note promising to pay 20 shillings to the bearer, is also disused for the present, and the sovereign supplies its place; but the name pound is still given to 20 shillings.

Also 6 Feet are 1 fathom .. 1 fth.
$69\frac{1}{2}$ Miles ......1 degree .. 1 deg. or 1°.
And a geographical mile is $\frac{1}{60}$ of a degree.

In the measurement of cloth or linen the following are also used:

$2\frac{1}{4}$ Inches are	1 nail	1 nl.
4 Nails	1 quarter (of a yard)	1 qr.
3 Quarters	1 Flemish ell	1 Fl. e.
5 Quarters	1 English ell	1 E. e.
6 Quarters	1 French ell	1 Fr. e.

(133.) MEASURES OF SURFACE, OR SUPERFICIES.

All surfaces are measured by square inches, square feet, &c.; the square inch being a square whose side is an inch in length, and so on.

144 *Square inches* are	1 *square foot*	1 sq. ft.
9 *Square feet*	1 *square yard*	1 sq. yd.
$30\frac{1}{4}$ Square yards	1 square pole	1 sq. p.
40 Square poles	1 rood	1 rd.
4 Roods	1 acre	1 ac.

N.B. *A square whose side is* $69\frac{4}{7}$ *yards is nearly an acre.*

(134.) MEASURES OF SOLIDITY OR CAPACITY *.

Cubes are solids having the figure of dice. A cubic inch is a cube each of whose sides is an inch, and so on.

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* Capacity is a term which cannot be better explained than by its use. When one measure holds more than another, it is said to be more capacious, or to have a greater capacity.

1728 Cubic inches	are	1 cubic foot	1 c. ft.
27 Cubic feet		1 cubic yard	1 c. yd.

This measure is not in general use, except in purely mathematical questions. In the measurements of different commodities various measures were used, until last year, when by an act of parliament the measures of capacity were all reduced to two. These are commonly called the Imperial Measures, and are as follows:

I. Measure of liquids, and of all dry goods which are not usually heaped up above the measure.

4 *Gills*	are	1 *pint*	1 pt.
2 *Pints*		1 *quart*	1 qt.
4 *Quarts*		1 *gallon*	1 gall.
2 *Gallons*		1 *peck*[*]	1 pk.
4 *Pecks*		1 *bushel*	1 bu.
8 *Bushels*		1 *quarter*	1 qr.
5 *Quarters*		1 *load*	1 ld.

The gallon in this measure is about 277·274 cubic inches; that is, very nearly 277¼ cubic inches.

II. The other Imperial Measure is applied to goods which it is customary to sell by *heaped measure*, and is as follows:

2 Gallons	1 peck
4 Pecks	1 bushel
3 Bushels	1 sack
12 Sacks	1 chaldron.

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[*] This measure and those which follow are used for dry goods only.

The gallon in this measure is about 704 cubic inches.

(135.) MEASURES OF WEIGHT.

There are several measures of weight still in use. The grain is common to all of them, and is the least of them all. The most common of all these measures, by which every thing is bought and sold except gold, silver, precious stones, and also medicines, is what is called the Avoirdupois weight, and is as follows:

$27\frac{11}{32}$ Grains	are	1 dram	1 dr.
16 *Drams*		1 *ounce*	1 oz.
16 *Ounces*		1 *pound*	1 lb.
28 *Pounds*		1 *quarter*	1 qr.
4 *Quarters*		1 *hundredweight*	1 cwt.
20 *Hundredweight*		1 *ton*	1 ton.

There are many terms made use of in different trades, which it is not worth while to recollect. Thus, in the wool trade, 14 lbs. is called a stone, 2 stone a tod, &c.

Gold, silver, and precious stones are weighed by the following table, which is called Troy Weight:

24 *Grains*	are	1 *pennyweight*
20 *Pennyweights*		1 *ounce*
12 *Ounces*		1 *pound.*

Medicines are weighed by the following table, which is called Apothecaries' Weight:

20 *Grains* are 1 *scruple*.... ℈ [*]
3 *Scruples*.... 1 *dram*...... ʒ
8 *Drams*...... 1 *ounce*..... ℥
12 *Ounces*...... 1 *pound*..... ℔

(136.) MEASURE OF TIME.

The day is the average time which elapses between noon and noon. Time is measured in the following way. The second, marked thus, 1″, or $\frac{1}{86400}$ of a day, is the lowest measure in use.

60 *Seconds* are 1 *minute*.... 1′
60 *Minutes*..... 1 *hour* ..... 1 h.
24 *Hours*...... 1 *day*...... 1 d.
7 *Days* .... 1 *week* ..... 1 wk.
365 *Days* ...... 1 *year*[†] .... 1 yr.

(137.) When any quantity is made up of several others expressed in different units, such as £1 . 14 . 6, or 2 cwt. 1 qr. 3 lbs., it is called a *compound quantity*. From these tables it is evident that any compound quantity of any substance is measured in several different ways. For example, the sum of money which we call five pounds four shillings is also 104 shillings, or 1248 pence, or 4992 farthings. It is very easy to reduce any quantity from one of these measurements to another; and the following examples will be sufficient to enable you to apply the same process to all sorts of quantities.

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[*] These are the characters used by physicians in writing prescriptions.

[†] Every fourth year has 366 days, and is called leap year. The additional day is the 29th of February, a month which, in every other year, has only 28 days. The next leap year is A.D. 1832.

I. How many farthings are there in £18 . 12 . $6\frac{3}{4}$[*]?

Since there are 20 shillings in a pound, there are, in £18, 18×20, or 360 shillings; therefore £18 . 12 is $360 + 12$, or 372 shillings. Since there are 12 pence in a shilling, in 372 shillings there are 372×12, or 4464 pence; and therefore in £1 . 18 . 6 there are $4464 + 6$, or 4470 pence.

Since there are 4 farthings in a penny, in 4470 pence there are 4470×4, or 17880 farthings; and therefore in £1 . 18 . $6\frac{3}{4}$ there are $17880 + 3$, or 17883 farthings. The whole of this process may be written as follows:

$$\begin{array}{llll}
£18\ .\ 12\ .\ 6\frac{3}{4} & & & \\
\underline{\ 20\qquad} & & & \\
360 + 12 = & 372 & & \\
 & \underline{\ 12\ } & & \\
 & 4464 + 6 = & 4470 & \\
 & & \underline{\ \ 4\ } & \\
 & & 17880 + 3 = & 17883
\end{array}$$

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[*] Farthings are never written but as parts of a penny. Thus three farthings, being $\frac{3}{4}$ of a penny, is written $\frac{3}{4}$, or ${}^{3}/_{4}$. One halfpenny may be written either as $\frac{2}{4}$ or $\frac{1}{2}$; the latter is most common.

II. In 17883 farthings, how many pounds, shillings, pence, and farthings are there?

Since 17883, divided by 4, gives the quotient 4470 and the remainder 3 (60), (131), 17883 farthings are 4470 pence and 3 farthings.

Since 4470, divided by 12, gives the quotient 372 and remainder 6, 4470 pence is 372 shillings and 6 pence.

Since 372, divided by 20, gives the quotient 18 and the remainder 12, 372 shillings is 18 pounds and 12 shillings.

Therefore 17883 farthings is 4470¾*d.*, which is 372*s.* 6¾*d.*, which is £18 . 12 . 6¾.

The process may be written as follows:

4)17883
12)4470 – 3
20)372 – 6
£18 . 12 . 6¾

EXERCISES.

A has £100 . 4 . 11½, and B has 64392 farthings. If A receives 1492 farthings, and B £1 . 2 . 3½, which will then have the most, and by how much?

How many pounds, ounces, pennyweights, and grains, Troy, are there in 663072 grains, in 9220 pennyweights, and in 2100907 ounces?

How many seconds of time are there in 1 day, 15 minutes, and 13 seconds; or in 1 d. 15′ 13″?

In 272 acres, how many square yards are there?

(138.) The same may be done where the number first expressed is fractional. For example, how many shillings and pence are there in $\frac{4}{15}$ of a pound? Now $\frac{4}{15}$ of a pound is $\frac{4}{15}$ of 20 shillings, $\frac{4}{15}$ of 20 is (99) $\frac{4\times 20}{15}$, or $\frac{4\times 4}{3}$, or $\frac{16}{3}$, or (87) $5\frac{1}{3}$ of a shilling. Now $\frac{1}{3}$ of a shilling is $\frac{1}{3}$ of 12 pence, or 4 pence. Therefore £$\frac{4}{15}$ = 5*s.* 4*d.*

Again, suppose it required to find what part of a pound is 6*s.* 8*d.* Since 6*s.* 8*d.* is 80 pence, and since the whole pound contains 20 × 12, or 240 pence, 6*s.* 8*d.* is made by dividing the pound into 240 parts, and taking 80 of them. It is therefore £$\frac{80}{240}$ (90), but $\frac{80}{240} = \frac{1}{3}$ (91). Therefore, 6*s.* 8*d.* = £$\frac{1}{3}$.

EXERCISES.

What number of hours, minutes, and seconds, are there in the following fractions of a day—$\frac{2}{5}$, $\frac{1}{70}$, $\frac{29}{44}$, ·12841, and ·00130?

How much of a hundredweight is there in 72 lbs. 2 oz. 15 dr.?

How much of a shilling is there in $\frac{2}{365}$ of a pound?

How much of 7 cwt. 14 lbs. is there in 2 oz. 13 dr.?

(139.) The rule for addition* of two compound quantities of the same sort will be evident from the following example. Suppose it required to add £192 . 14 . $2\frac{1}{2}$ to £64 . 13 . $11\frac{3}{4}$. The sum of these two is the whole of that which arises from adding their several parts. Now

---

* Before reading this article and the next, you should read carefully again articles (25) and (34).

$$\begin{array}{llll}
\frac{3}{4}d. + & \frac{1}{2}d. = & \frac{5}{4}d. = & \pounds 0 \;.\; 0 \;.\; 1\frac{1}{4} \quad (137) \\
11d. + & 2d. = & 13d. = & \quad 0 \;.\; 1 \;.\; 1 \\
13s. + & 14s. = & 27s. = & \pounds 1 \;.\; 7 \;.\; 0 \\
\pounds 64 + & \pounds 192 = & & \pounds 256 \;.\; 0 \;.\; 0 \\
\hline
\end{array}$$

The sum of all of which is, £257 . 8 . $2\frac{1}{4}$

The whole of this may be done at once, and written as follows:—

$$\begin{array}{r}
\pounds 192 \;.\; 14 \;.\; 2\frac{1}{2} \\
64 \;.\; 13 \;.\; 11\frac{3}{4} \\
\hline
\pounds 257 \;.\; 8 \;.\; 2\frac{1}{4}
\end{array}$$

Begin by adding together the farthings, and reduce the result to pence and farthings. Set down the last only, and carry the first to the line of pence, and add the pence in both lines to it. Reduce these pence to shillings and pence; set down the last only, and carry the first to the line of shillings, and so on. The same method must be followed when the quantities are of any other sort; and if the tables are kept in memory the process will be easy.

(140.) Subtraction is performed on exactly the same principles as in (33), namely, that you do not alter the difference of two quantities by adding the same quantity to both. Suppose it required to subtract £19 . 13 . $10\frac{3}{4}$ from £24 . 5 . $7\frac{1}{2}$. Write these quantities under one another thus—

£24 . 5 . $7\frac{1}{2}$
19 . 13 . $10\frac{3}{4}$

Since $\frac{3}{4}$ cannot be taken from $\frac{1}{2}$ or $\frac{2}{4}$, add 1*d.* to both quantities, which will not alter their difference; or, which is the same thing, add 4 farthings to the first and 1*d.* to the second. The pence and farthings in the two lines then stand thus—$7\frac{6}{4}$*d.* and $11\frac{3}{4}$*d.* Now subtract $\frac{3}{4}$ from $\frac{6}{4}$, and the difference is $\frac{3}{4}$, which must be written under the farthings. Again, since 11*d.* cannot be subtracted from 7*d.* add 1*s.* to both quantities by adding 12*d.* to the first and 1*s.* to the second. The pence then are in the first line 19, in the second 11, and the difference is 8, which write under the pence. Since the shillings in the lower line were increased by 1, there are now 14*s.* in the lower and 5*s.* in the upper one. Add 20*s.* to the upper and £1 to the lower line, and the subtraction of the shillings in the second from those in the first leaves 11*s.* Again, there are now £20 in the lower and £24 in the upper line, the difference of which is £4; therefore the whole difference of the two sums is £4 . 11 . $8\frac{3}{4}$. If we write down the two sums with all the additions which have been made, the process will stand thus—

£24 . 25 . $19\frac{6}{4}$
20 . 14 . $11\frac{3}{4}$

Difference £ 4 . 11 . $8\frac{3}{4}$

You must bear constantly in mind that, by adding the same quantity to two other quantities, you do not alter their difference.

(141.) The same method may be applied to any of the quantities in the tables. The following is another example:—

From 7 cwt. 2 qrs. 21 lbs. 14 oz.
Subtract 2 cwt. 3 qrs. 27 lbs. 12 oz.

After alterations have been made similar to those in the last article, the question becomes—

From 7 cwt. 6 qrs. 49 lbs. 14 oz.
Subtract 3 cwt. 4 qrs. 27 lbs. 12 oz.

The difference is 4 cwt. 2 qrs. 22 lbs. 2 oz.

In this example, and almost every other, the process may be a little shortened in the following way. Here we do not subtract 27 lbs. from 21 lbs., which is impossible, but we increase 21 lbs. by 1 qr. or 28 lbs., and then subtract 27 lbs. from the sum. It is shorter, and would be the same thing, first to subtract 27 lbs. from 1 qr. or 28 lbs. and add the difference to 21 lbs.

(142.) EXERCISES.

A man has the following sums to receive— £193 . 14 . 11¼., £22 . 0 . 6¾., £6473 . 0 . 0., and £49 . 14 . 4½; and the following debts to pay— £200 . 19 . 6¼, £305 . 16 . 11, £22, and

£19 . 6 . 0½. How much will remain after paying the debts?

There are four towns, in the order A, B, C, and D. If a man can go from A to B in 5 h. 20′ 33″, from B to C in 6 h. 49′ 2″, and from A to D in 19 h. 0′ 17″, how long will he be in going from C to D?

(143.) In order to perform the process of multiplication, you must recollect that, as in (45), if a quantity be divided into several parts, and each of these parts be multiplied by a number, and the products be added, the result is the same as would arise from multiplying the whole quantity by that number.

It is required to multiply £7 . 13 . 6¼ by 13. The first quantity is made up of 7 pounds, 13 shillings, 6 pence, and 1 farthing. And

1 farth.	× 13 is	13 farth.	or £0 . 0 . 3¼	(137)
6 pence	× 13 is	78 pence,	or 0 . 6 . 6	
13 shill.	× 13 is	169 shill.	or 8 . 9 . 0	
7 pounds	× 13 is	91 pounds,	or 91 . 0 . 0	
The sum of all these is			£99 . 15 . 9¼	

which is therefore £7 . 13 . 6¼ × 13.

This process is usually written as follows:—

$$\begin{array}{r} \pounds 7 \;.\; 13 \;.\; 6\tfrac{1}{4} \\ 13 \\ \hline \pounds 99 \;.\; 15 \;.\; 9\tfrac{1}{4} \end{array}$$

(144.) Division is performed upon the same principle as in (61), viz., that if a quantity be divided into any number of parts, and each part be divided by any number, the different quotients added together will make up the quotient of the whole quantity divided by that number. Suppose it required to divide £99 . 15 . 9¼ by 13. Since 99 divided by 13 gives the quotient 7 and the remainder 8, the quantity is made up of £13 × 7, or £91 and £8 . 15 . 9¼. The quotient of the first, 13 being the divisor, is £7: it remains to find that of the second. Since £8 is 160*s.*, £8 . 15 . 9¼ is 175*s.* 9¼*d.*, and 175 divided by 13 gives the quotient 13 and the remainder 6; that is, 175*s.* 9¼*d.* is made up of 169*s.* and 6*s.* 9¼*d.*, the quotient of the first of which is 13*s.*, and it remains to find that of the second. Since 6*s.* is 72*d.*, 6*s.* 9¼*d.* is 81¼*d.*, and 81 divided by 13 gives the quotient 6 and remainder 3; that is, 81¼*d.* is 78*d.* and 3¼*d.*, of the first of which the quotient is 6*d.* Again, since 3*d.* is $\frac{12}{4}$ or 12 farthings, 3¼*d.* is 13 farthings, the quotient of which is 1 farthing, or ¼, without remainder. We have then divided £99 . 15 . 9¼ into four parts, each of which is divisible by 13, viz., £91, 169*s.*, 78*d.*, and 13 farthings; so that the thirteenth part of this quantity is £7 . 13 . 6¼. The whole process may be written down as follows, and the same sort of process may be applied to the exercises which follow:

```
     £   s.  d.  £  s.  d.
  13)99  15  9¼(7  13  6¼
     91
     ——
      8
     20
     ——
    160+15=175
            13
            ——
             45
             39
             ——
              6
             12
             ——
             72+9=81
                  78
                  ——
                   3
                   4
                  ——
                  12+1=13
                       13
                       ——
                        0
```

Here each of the numbers 99, 175, 81, and 13 is divided by 13 in the usual way, though the divisor is only written before the first of them.

EXERCISES.

Multiply 2 cwt. 1 qr. 21 lbs. 7 oz. by 53
and 2 d. 4 h. $3'$ $27''$ by 109
Divide 144 gall. 2 qts. 1 pt. by 8
and £24 . 5 . 4¾ by 3

(145.) Suppose it required to find how many times 1*s.* 4¼*d.* is contained in £3 . 19 . 10¾. The best way to do this is to find the number of farthings in each. By (137), in the first there are 65, and, in the second, 3835 farthings. Now 3835 contains 65, 59 times; and therefore the second quantity is 59 times as great as the first.

EXERCISES.

How many times does

6 cwt. 2 qrs. contain 1 qr. 14 lbs. 1 oz.?
and 1 d. 2 h. $0'$ $47''$ contain $3'$ $46''$?

If 2 cwt. 3 qrs. 1 lb. cost £150 . 13 . 10, how much does 1 lb. cost?

A grocer mixes 2 cwt. 15 lbs. of sugar at 11*d.* a pound with 14 cwt. 3 lbs. at 5*d.* a pound. At how much per pound must he sell the mixture so as not to lose by mixing them?

(146.) There is a convenient method of multiplication called Practice, which I now proceed to explain. Suppose I ask how much do 153 ton cost, if each ton cost £2 . 15 . 7¾? It is plain that if this sum be multiplied by 153, the product is the price of the whole. But this is also evident, that, if I buy 153 ton at £2 . 15 . 7½ each ton, payment may be made by first putting down £2 for each ton, then 10*s.* for each, then 5*s.*, then 6*d.*, and then 1½*d.* These sums together make up £2, 15*s.* 7½*d.*, and the reason for this separation of £2 . 15 . 7½ into different parts will be soon apparent. The process may now be carried on as follows:

1. 153 ton, at £2 each ton, will cost	£306	0	0
2. Since 10*s.* is £½, 153 ton, at 10*s.* each, will cost £$\frac{153}{2}$, which is	76	10	0
3. Since 5*s.* is ½ of 10*s.*, 153 ton, at 5*s.*, will cost half as much as the same number at 10*s.* each, that is, ½ of £76, 10*s.*, which is .	38	5	0
4. Since 6*d.* is $\frac{1}{10}$ of 5*s.*, 153 ton, at 6*d.* each, will cost $\frac{1}{10}$ of what the same number costs at 5*s.* each, that is, $\frac{1}{10}$ of £38, 5*s.*, which is	3	16	6
5. Since 1½ or 3 halfpence is ¼ of 6*d.* or 12 halfpence, 153 ton, at 1½*d.* each, will cost ¼ of what the same number costs at 6*d.* each, that is, ¼ of £3 . 16 . 6, which is	0	19	1½
The sum of all these quantities is .	425	10	7½

which is, therefore, £2 . 15 . 7½ × 153.

The whole process may be written down as follows:

	£153	0	0	or what 153 ton would cost at	£1	per	ton.
£2 is 2 × £1	306	0	0		2	0	0
10s. is ½ of £1	76	10	0		0	10	0
5s. is ½ of 10s.	38	5	0		0	5	0
6d. is ⅒ of 5s.	3	16	6		0	0	6
1½d. is ¼ of 6d.	0	19	1½		0	0	1½
	£425	10	7½		£2	15	7½

ANOTHER EXAMPLE.

What do 1735 lbs. cost at 9s. 10¾d. per lb.? The price 9s. 10¾d. is made up of 5s., 4s., 10d., ½d., and ¼d.; of which 5s. is ¼ of £1, 4s. is ⅕ of £1, 10d. is ⅙ of 5s., ½d. is $\frac{1}{20}$ of 10d., and ¼d. is ½ of ½d. Follow the same method as in the last example, which gives the following:

	£1735	0	0	or what 1735 lbs. would cost at	£1	per	lb.
5s. is ¼ of £1	433	15	0		0	5	0
4s. is ⅕ of £1	347	0	0		0	4	0
10d. is ⅙ of 5s.	72	5	10		0	0	10
½d. is $\frac{1}{20}$ of 10d.	3	12	3½		0	0	0½
¼d. is ½ of ½d.	1	16	1¾		0	0	0¼
by addition	£858	9	3¼		£0	9	10¾

In all cases the price must first be divided into a number of parts, each of which is a simple fraction* of some one which goes before. No rule can be given for doing this, but a little practice will enable you immediately to find out the best method for each case. When that is done, you must find how much the whole quantity would cost if each of these parts were the price, and then add these quantities together.

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* Any part of a unit, which is a fraction of it, whose numerator is unity, is generally called an *aliquot part* of that unit. Thus 2s. and 10s. are both aliquot parts of a pound, being £⅒ and £½.

EXERCISES.

What is the cost of

243 cwt. at £14 . 17 . 11¾ per cwt.?
169 bu. at £2 . 1 . 3¼ per bushel?
273 qrs. at 19*s.* 2*d.* per quarter?
2627 sacks at £22 . 7 . 8½ per sack?

(147.) Throughout this section, it must be observed, that the same rules can be applied to cases where the quantities given are expressed in common or decimal fractions, instead of the measures explained in the tables. The following exercises on the rules of this section will serve as examples.

What is the price of 272·3479 cwt. at £2 . 1 . 3½ per cwt.?

In ·03841 of a pound sterling, how many shillings, pence, and farthings, are there?

·253 of a quarter is added to 12·237 lbs.: what decimal of a cwt. is the sum?

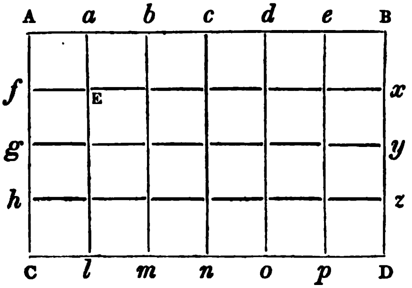
How many pounds, shillings, and pence, will 279·301 acres let for, if each acre lets for £3·1076?

What is $\frac{2}{17}$ of £1·906 in shillings, pence, and farthings?

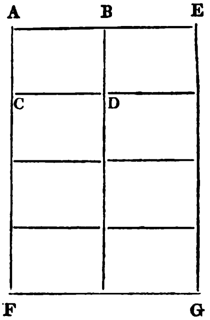
What does $\frac{1}{4}$ of $\frac{3}{13}$ of 17 bush. cost at $\frac{1}{5}$ of $\frac{2}{3}$ of £17, 14*s*. per bushel?

(148.) The following connexion between the measures of length and the measures of surface is the foundation of the application of arithmetic to geometry.

Suppose an oblong figure ABCD, such as is here drawn (which is called a *rectangle* in geometry), has



the side AB 6 inches in length, and the side AC 4 inches. Divide AB and CD (which are equal) each into 6 inches by the points a, b, c, d, e, &c.; and AC and BD (which are also equal) into 4 inches by the points f, g, h, x, y, and z. Join a and l, b and m, &c. and f and x, &c. Then the figure ABCD is divided into a number of squares, since a square figure only differs from a rectangle in having its sides equal to one another, and therefore A$af$E is a square, since Aa is of the same length as Af, both being 1 inch. There are also four rows of these squares, with six squares in each row; that is, there are 6 × 4, or 24 squares altogether. Each of these squares has its sides one inch in length, and is what was called in (133) *a square inch.* By the same reasoning, if one side had contained 6 *yards*, and the other 4 *yards*, the surface would have contained 6 × 4 *square yards*, and so on.



(149.) Let us now suppose that the sides of ABCD, instead of being a whole number of inches, contain some inches and a fraction. For example, let AB be $3\frac{1}{2}$ inches, or (96) $\frac{7}{2}$ of an inch, and let AC contain $2\frac{1}{4}$ inches, or $\frac{9}{4}$ of an inch. Draw AE twice as long as AB, and AF four times as long as AC, and complete the rectangle AEFG. The rest of the figure needs no description. Then since AE is twice AB, or twice $\frac{7}{2}$ inches, it is 7 inches. And since AF is four times AC, or four times $\frac{9}{4}$ inches, it is 9 inches. Therefore the whole rectangle AEFG contains, by the last article, 7 × 9, or 63 square inches. But the rectangle AEFG contains 8 rectangles, all of the same figure as ABCD; and therefore ABCD is one-eighth part of AEFG, and contains $\frac{63}{8}$ square inches. But $\frac{63}{8}$ is made by multiplying $\frac{9}{4}$ and $\frac{7}{2}$ together (100). From this and the last article it appears that, whether the sides of a rectangle be a whole or a fractional number of inches, the number of square inches in its surface is the product of the numbers of inches in its sides. The square itself is a rectangle whose sides are all equal, and therefore the number of square inches which a square contains is found by multiplying the number of inches in its side by itself. For example a square whose side is 13 inches in length contains 13 × 13, or 169 square inches.

(150.) EXERCISES.

What is the content, in square feet and inches, of a room whose sides are 42 ft. 5 inch. and 31 ft. 9 inch.? and supposing the piece from which its carpet is taken to be three quarters of a yard in breadth, what length of it must be cut off to form a carpet for the room?

The sides of a rectangular field are 253 yards and a quarter of a mile; how many acres does it contain?

What is the difference between 18 square miles, and a square of 18 miles long?

SECTION II.

RULE OF THREE.

(151.) Suppose it required to find what 156 yards will cost, if 22 yards cost 17*s.* 4*d.* This quantity, reduced to pence, is 208*d.*; and if 22 yards cost 208*d.* each yard costs $\frac{208}{22}$*d.* But 156 yards cost 156 times the price of one yard, and therefore cost $\frac{208}{22} \times 156$ pence, or $\frac{208 \times 156}{22}$ pence (99). Again, if $25\frac{1}{2}$ French francs are 20 shillings sterling, how many francs are in £20, 15*s.*? Since $25\frac{1}{2}$ francs are 20 shillings, twice the number of francs must be twice the number of shillings; that is, 51 francs is 40 shillings, and one shilling is the fortieth part of 51 francs, or $\frac{51}{40}$ francs. But £20. 15*s.* contains 415 shillings (137); and since 1 shilling is $\frac{51}{40}$ francs, 415 shillings is $\frac{51}{40} \times 415$ francs, or (99) $\frac{51 \times 415}{40}$ francs.

(152.) Such questions as the last two belong to the most extensive rule in Commercial Arithmetic, which is called the RULE OF THREE, because in it three quantities are given, and a fourth is required to be found. From both the examples given the following rule may be deduced, and the same reasoning will show that the rule applies to other similar cases.

It must be observed, that in all these questions there are two quantities which are of the same sort, and a third of another sort, of which last sort the answer must be. Thus, in the first question there are 22 and 156 yards and 208 pence, and the thing required to be found is a number of pence. In the second question there are 20 and 415 shillings and $25\frac{1}{2}$ francs, and what is to be found is a number of francs. Write the three quantities in a line, putting that one last which is the only one of its kind, and that one first which is connected with the last in the question[*]. Put the third quantity in the middle. In the first question the quantities will be placed thus—

22 yds. 156 yds. 17*s.* 4*d.*

In the second question they will be placed thus—

20*s.* £20. 15*s.* 25½ francs.

Reduce the first and second quantities, if necessary, to quantities of the same denomination. Thus, in the second question, £20. 15*s.* must be reduced to shillings (137). The third quantity may also be reduced to any other denomination, if convenient; or the first and third may be multiplied by any quantity we please, as was done in the second question; and on looking at the answer in (151), and at (91), you will see that no change is made by that multiplication. When this has been done, multiply the second and third quantities together, and divide by the first. The result is a quantity of the same sort as the third in the line, and is the answer required. Thus, to the first question the answer is (151) $\frac{208 \times 156}{22}$ pence, or, which is the same thing, $\frac{17s.\ 4d. \times 156}{22}$.

---

[*] This generally comes in the same member of the sentence. In some cases the ingenuity of the student must be employed in detecting it. The reasoning of (151) is the best guide.

(153.) The whole process in the first question is as follows*:

Yds. Yds. *s.* *d.*
22 : 156 :: 17 . 4
12

208 pence.
156

1248
1040
208

22)32448($1474\frac{3}{4}$*d.* and $\frac{14}{22}$ or $\frac{7}{11}$ of a farthing, or (137) £6 . 2 . $10\frac{3}{4}$ $\frac{7}{11}$.
22

104
88

164
154

108
88

20
(144) 4

80
66

14

---

* It is usual to place points, in the manner here shown, between the quantities.

The question might have been solved without reducing 17*s.* 4*d.* to pence, thus:

Yds. Yds. *s.* *d.*
22 : 156 :: 17 . 4
156 (143)

22)£165 . 4 . 0(£6 . 2 . $10\frac{3}{4}\frac{7}{11}$ (144)
132

3 × 20 + 4 = 64
44

20 × 12 = 240
22

20 × 4 = 80
66

14

The student must learn by practice which is the most convenient method for any particular case, as no rule can be given.

(154.) It may happen that the three given quantities are all of one denomination; nevertheless, it will be found that two of them are of one, and the third of another sort. For example, take this question, What must an income of £400 pay towards an income-tax of 4*s.* 6*d.* in the pound? Here the three given quantities are £400. 4*s.* 6*d.*, and £1, which are all of the same species, viz. money. Nevertheless, the first and third are income, the second is a tax, and the answer is also a tax; and therefore, by (152), the quantities must be placed thus:

£1 : £400 :: 4*s*. 6*d*.

(155.) The following exercises either depend directly upon this rule, or can be shown to do so by a little consideration. There are many questions of the sort, which will require some exercise of ingenuity before the method of applying the rule can be found.

EXERCISES.

If 15 cwt. 2 qrs. cost £198. 15*s*. 4*d*., what does 1 qr. 22 lbs. cost?

If a horse go 23 miles in 3 h. 26′ 12″, how long will he be in going 14 m. 3 fur. 27 yds.?

Two persons, A and B, are bankrupts, and owe exactly the same sum; A can pay 15*s*. $4\frac{1}{2}$*d*. in the pound, and B only 7*s*. $6\frac{3}{4}$*d*. At the same time, A has in his possession £1304. 17*s*. more than B. What do the debts of each amount to?

For every $12\frac{1}{2}$ acres which one country contains, a second contains $56\frac{1}{4}$. The second country contains 17300 square miles. How much does the first contain?—Again, for every 3 people in the first, there are 5 in the second; and there are in the first 27 people on every 20 acres. How many are there in the second country?

If $42\frac{1}{2}$ yards of cloth, 15 inches wide, cost £59. 14*s*. 2*d*., how much will $118\frac{1}{3}$ yards cost, if the width be 1 yard?

(156.) Suppose the following question asked: How long will it take 15 men to do that which 45 men can finish in 10 days? It is evident that one man would take 450 or 45 × 10 days to do the same thing, and that 15 men would do it in one-fifteenth part of the time which it employs one man, that is, in $\frac{450}{15}$ or 30 days. By this and similar reasoning the following questions can be solved.

EXERCISES.

If 15 oxen eat an acre of grass in 12 days, how long will it take 26 oxen to eat 14 acres?

If 22 masons build a wall 5 feet high in 6 days, how long will it take 43 masons to do the same?

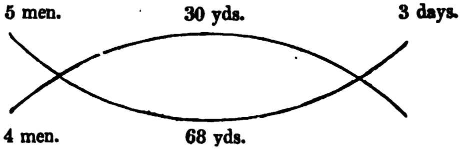
(157.) The questions in the preceding article, though their solution is too simple to require that a rule should be stated for it, form a part of a more general class of questions, whose solution is called the Double Rule of Three, but which might, with more correctness, be called the rule of *Five*, since five quantities are given, and a sixth is to be found. The following is an example: If 5 men can make 30 yards of cloth in 3 days, how long will it take 4 men to make 68 yards? The first thing to be done is to find out, from the first part of the question, how long it will take one man to make one yard. Now, since one man, in three days, will do the fifth part of what 5 men can do, he will make in 3 days $\frac{30}{5}$, or 6 yards. He will, therefore, make one yard in $\frac{3}{6}$ or in $\frac{3\times5}{30}$ of a day.

From this we are to find how long it will take 4 men to make 68 yards. Since one man makes a yard in $\frac{3\times 5}{30}$ of a day, he will make 68 yards in $\frac{3\times 5}{30}\times 68$ days, or (98) in $\frac{3\times 5\times 68}{30}$ days; and 4 men will do this in one-fourth of the time, that is (105), in $\frac{3\times 5\times 68}{30\times 4}$ days, or in $8\frac{1}{2}$ days.

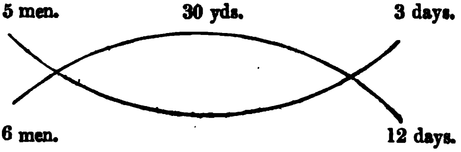
Again, suppose the question to be, If 5 men can make 30 yards in 3 days, how much can 6 men do in 12 days? Here we must first find how much one man can do in one day, which appears, on reasoning similar to that in the last example, to be $\frac{30}{3\times 5}$ yards. Hence 6 men, in one day, will make $\frac{6\times 30}{5\times 3}$ yards, and in 12 days will make $\frac{12\times 6\times 30}{5\times 3}$, or 24 yards.

From these examples the following rule may be drawn. Write the given quantities in two lines, keeping quantities of the same sort under one another, and those which are connected with each other, in the same line. In the two examples above given, the quantities must be written thus:

First Example.



Second Example.



Draw a curve through the middle of each line, and the extremities of the other. There will be three quantities on one curve, and two on the other. Divide the product of the three by the product of the two, and the quotient is the answer to the question.

If necessary, the quantities in each line must be reduced to more simple denominations (137), as was done in the common Rule of Three (151).

EXERCISES.

If 6 horses can, in 2 days, plough 17 acres, 2 poles, how many acres will 93 horses plough in 4½ days?

If 20 men, in 3¼ days, can dig 7 rectangular fields, the sides of each of which are 140 and 150 yards, how long will 37 men be in digging 53 fields, the sides of each of which are 90 and 125½ yards?

SECTION III.

INTEREST, &c.

(158.) In all the questions which are contained in this section, almost the only process which will be employed is the taking a fractional part of a sum of money. Though this has been done before in several cases, I subjoin some new examples. Suppose it required to take away 7 parts out of 40 from £16, that is, to divide £16 into 40 equal parts, and take 7 of them. One of these parts is £$\frac{16}{40}$, and 7 of them are $\frac{16}{40}\times 7$, or $\frac{16\times 7}{40}$ pounds (99). The process may be written as follows:

```
                £16
                  7
                ———
(144)       40)112(£2 . 16s.
                80
                ———
                32
                 20
                ———
                640
                40
                ———
                240
                240
                ———
                  0
```

Suppose it required to take 13 parts out of a hundred from £56. 13*s*. 7½*d*.

£ *s.* *d.*
56 . 13 . 7½
(143) 13

100)736 . 17 . 1½(£7 . 7 . 4¼ $\frac{41}{50}$
700

(144) $36\times20+17=737$
700

$37\times12+1=445$
400

$45\times4+2=182$
100

82

Suppose it required to take 2½ out of a hundred from £3 . 12*s.* The result, by the same rule, is $\frac{£3.\ 12s. \times 2\frac{1}{2}}{100}$ or (105) $\frac{£3.\ 12s. \times 5}{200}$; so that taking 2½ out of a hundred is the same as taking 5 parts out of 200.

EXERCISES.

Take 7⅓ parts out of 53 from £1. 10*s.*

Take 5 parts out of 100 from £107. 13*s.* 4¾*d.*

£56. 3*s.* 2*d.* is equally divided among 32 persons. How much does the share of 23 of them exceed that of the rest?

(159.) It is usual, in mercantile business, to mention the fraction which one sum is of another, by saying how many parts out of a hundred must be taken from the second in order to make the first. Thus, instead of saying that £16. 12*s.* is the half of £33. 4*s.*, it is said that the first is 50 per cent. of the second. Thus £5 is $2\frac{1}{2}$ per cent. of 200; because, if 200 be divided into 100 parts, $2\frac{1}{2}$ of those parts are £5. Suppose it asked, how much per cent. is 23 parts out of 56 of any sum? Now 23 parts out of 56 is $\frac{23}{56}$ of the whole sum (90), which is (91) $\frac{2300}{5600}$ or $\frac{\frac{2300}{56}}{100}$ (105), which is $\frac{2300}{56}$ parts out of a hundred, or $41\frac{1}{14}$ per cent.

Similarly 16 parts out of 18 is $\frac{16 \times 100}{18}$, or $88\frac{8}{9}$ per cent.

2 parts out of 5 is $\frac{2 \times 100}{5}$, or 40 per cent.

From which the method of reducing other fractions to the rate per cent. is evident.

Suppose it asked, how much per cent. is £6. 12*s.* 2*d.* of £12. 3*s.*? Since the first contains 1586*d.* and the second 2916*d.*, the first is 1586 out of 2916 parts of the second; that is, by the last rule, it is $\frac{158600}{2916}$ or $54\frac{1136}{2916}$, or £54. 7*s.* $9\frac{1}{2}$*d.* per cent. very nearly.

EXERCISES.

How much per cent. is $198\frac{1}{4}$ out of 233 parts?

Goods which are bought for £193. 12*s.* are sold for £216. 13*s.* 4*d.*; how much per cent. has been gained by them?

A sells goods for B to the amount of £230. 12*s.*

and is allowed a commission* of 3 per cent.; what does that amount to?

A stockbroker buys £1700 stock, brokerage being at £$\frac{1}{8}$ per cent.; what does he receive?

A ship whose value is £15423 is insured at 19$\frac{2}{3}$ per cent.; what does the insurance amount to?

(160.) INTEREST is money paid for the use of other money, and is always a percentage upon the sum lent. It may be paid either yearly, half-yearly, or quarterly; but when it is said that £100 is lent at 4 per cent., it must be understood to mean 4 per cent. per annum; that is, that 4 pounds are paid every year for the use of £100.

The sum lent is called the *principal*, and the interest upon it is of two kinds. If the borrower pays the interest as soon as from the agreement it becomes due, it is evident that he has to pay the same sum every year, and that the whole of the interest which he has to pay in any number of years is one year's interest multiplied by the number of years. But if he does not pay the interest at once, but keeps it in his hands until he returns the principal, he will then have more of his creditor's money in his hands every year, and will have to pay interest upon each year's interest for the time during which he keeps it after it becomes due. In the first case, the interest is called *simple*, and in the second *compound*. The interest and principal together are called the *amount*.

---

* Commission is what is allowed by one merchant to another for buying or selling goods for him, and is usually a percentage on the whole sum employed. Brokerage is an allowance similar to commission, under a different name, principally used in the buying and selling of stock in the funds.

Insurance is a percentage paid to those who engage to make good to the payers any loss they may sustain by accidents from fire, storms, or death, according to the agreement, up to a certain amount which is named, and is a percentage upon the sum named. Tare, trett, and cloff, are allowances made in selling goods by wholesale, for the weight of the boxes or barrels which contain them, waste, &c.; and are usually either the price of a certain number of pounds of the goods for each box or barrel, or a certain allowance on each cwt.

(161.) What is the interest of £1049. 16*s*. 6*d*. for 6 years and one-third, at $4\frac{1}{2}$ per cent.? This interest must be $6\frac{1}{3}$ times the interest of the same sum for one year, which (158) is found by multiplying the sum by $4\frac{1}{2}$ and dividing by 100. The process is as follows:

$$
\begin{array}{lr|l}
(146) & & £1049 \;.\; 16 \;.\; 6 \\
\hline
 & 4 & 4199 \;.\; 6 \;.\; 0 \\
 & \frac{1}{2} & 524 \;.\; 18 \;.\; 3 \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
(68) & 100)47{,}24 \;.\; 4 \;.\; 3(£47 \;.\; 4 \;.\; 10\tfrac{11}{100} \\
 & 20 \\
 & \overline{4{,}84^{*}} \\
(144) & 12 \\
 & \overline{10{,}11\,\dagger}
\end{array}
$$

---

* Here the 4*s*. from the dividend is taken in.

† Here the 3*d*. from the dividend is taken in.

$$\begin{array}{rrrrl}
 & \text{£}47\ . & 4\ . & 10\tfrac{11}{100} & \text{Interest for one year.}\\
 & & & 6 & \\ \hline
 & 283\ . & 9\ . & 0\tfrac{66}{100} & \\
\tfrac{1}{3} & 15\ . & 14\ . & 11\tfrac{37}{100} & \\ \hline
 & \text{£}299\ . & 4\ . & 0\tfrac{3}{100} & \text{Interest for } 6\tfrac{1}{3} \text{ years.}
\end{array}$$

EXERCISES.

What is the interest of £105. 6*s*. 2*d*., for 19 years and 7 weeks, at 3 per cent.?

What is the difference between the interest of £50. 19*s*., for 7 years, at 3 per cent., and for 8 years at 2½ per cent.?

Show that the interest of any sum for 9 years, at 4 per cent., is the same as that of the same sum for 4 years at 9 per cent.

(162.) In order to find the interest of any sum at compound interest, it is necessary to find the amount of the principal and interest at the end of every year; because in this case (160) it is the whole of the amount of both principal and interest, at the end of the first, for which interest is paid during the second year. Suppose, for example, it is required to find the interest, for 3 years, on £100, at 5 per cent. compound interest. The following is the process:

	£100	First principal.
	5	First year's interest.
	105	Amount at the end of the first year.
(161)	5 . 5	Interest for the second year on £105.
	110 . 5	Amount at the end of the second year.
	5 . 10 . 3	Interest due for the third year.
	115 . 15 . 3	Amount at the end of three years.
	100 . 0 . 0	First principal.
	15 . 15 . 3	Interest gained in the three years.

When the number of years is great, and the sum considerable, this process is very troublesome; on which account tables are constructed which show the amount of one pound, for different numbers of years, at different rates of interest. To make use of these tables in the present example, look into the column headed "5 per cent.;" and opposite to the number 3, in the column headed "Number of years," you will find 1·157625; meaning that £1 will become £1·157625 in 3 years. Now £100 must become 100 times as great; and 1·157625 × 100 is 115·7625 (123); but (137) £·7625 is 15*s.* 3*d.*; therefore the whole amount of £100 is £115. 15*s.* 3*d.*, as before.

(163.) Suppose that a sum of money has lain at simple interest 4 years, at 5 per cent., and has, with its interest, amounted to £350: it is required to find what the sum was at first. Now, whatever this sum was, if we suppose it divided into 100 parts, 5 of those parts were added every year for 4 years, as interest; that is, 20 of those parts have been added to the first sum to make £350. If, therefore, £350 be divided into 120 parts, 100 of those parts are the principal which we want to find, and 20 parts are interest upon it; that is, the principal is £$\frac{100 \times 350}{120}$, or £291 . 13 . 4.

(164.) Suppose that A is engaged to pay B £350 at the end of four years from this time, and that it is agreed between them that the debt shall be paid immediately; suppose, also, that money can be employed at 5 per cent. simple interest;—it is plain that A ought not to pay to B the whole sum, £350; because, if he did, he would lose 4 years' interest of the money, and B would gain it. It is fair, therefore, that he should only pay to B as much as will, *with interest*, amount in four years to £350, that is (163), £291. 13*s.* 4*d.* Therefore £58. 6*s.* 8*d.* must be struck off the debt in consideration of its being paid before the time. This is called DISCOUNT; and £291. 13*s.* 4*d.* is called the *present value* of £350 due four years hence, discount being at 5 per cent. The rule for finding the present value of a sum of money from (163) is, Multiply the sum by 100, and divide the product by 100, increased by the product of the rate per cent. and number of years.

EXERCISES.

What is the discount on a bill of £138. 14*s*. 4*d*. due 2 years hence, discount being at $4\frac{1}{2}$ per cent.?

What is the present value of £1031. 17*s*. due 12 years hence, interest being at $6\frac{1}{2}$ per cent.?

(165.) Suppose it required to divide £100 among three persons in such a way that their shares may be as 6, 5, and 9; that is, so that for every £6 which the first has, the second may have £5, and the third £9. It is plain that if we divide the £100 into $6+5+9$, or 20 parts, the first must have 6 of those parts, the second 5, and the third 9. Therefore (158) their shares are respectively, £$\frac{100\times6}{20}$, £$\frac{100\times5}{20}$, and £$\frac{100\times9}{20}$, or £30, £25, and £45.

EXERCISES.

Divide £394. 12*s*. among four persons, so that their shares may be as 1, 6, 7, and 19.

Divide £20 among 6 persons, so that the share of each may be as much as those of all who come before put together.

(166.) When two or more persons employ their money together, and gain or lose a certain sum, it is evidently not fair that the gain or loss should be equally divided among them all, unless each contributed the same sum. Suppose, for example, A contributes twice as much as B, and they gain £15, A ought to gain twice as much as B; that is, if the whole gain be divided into 3 parts, A ought to have two of them and B one, or A should gain £10 and B £5. Suppose that A, B, and C engage in an adventure in which A embarks £250, B £130, and C £45. They gain £1000. How much of it ought each to have? Each one ought to gain as much for £1 as the others. Now since there are 250 + 130 + 45, or 425 pounds, embarked, which gain £1000, for each pound there is a gain of £$\frac{1000}{425}$. Therefore A should gain $\frac{1000\times 250}{425}$ pounds, B should gain $\frac{1000\times 130}{425}$ pounds, and C $\frac{1000\times 45}{425}$ pounds. On these principles, by the process in (158), the following questions may be answered.

A ship is to be insured, in which A has ventured £1928. 14*s.* and B £4963. 16*s.* The expense of insurance is £474. 10*s.* 2*d.* How much ought each to pay of it?

A loss of £149 is to be made good by three persons, A, B, and C. Had there been a gain, A would have gained 4 times as much as B, and C twice as much as A and B together. How much of the loss must each bear?

(167.) It may happen that several individuals employ several sums of money together for different times. In such a case, unless there is a special agreement to the contrary, it is right that the more time a sum is employed the more profit should be made upon it. If, for example, A and B employ the same sum for the same purpose, but A's money is employed twice as long as B's, A ought to gain twice as much as B. The principle is, that one pound employed for one month, or one year, ought to give the same return to each. Suppose, for example, that A employs £3 for 6 months, B £4 for 7 months, and C £12 for two months, and the gain is £100,—how much ought each to have of it? Now, since A employs £3 for 6 months, he must gain 6 times as much as if he employed it one month only; that is, as much as if he employed £6×3, or £18, for one month: also B gains as much as if he had employed £4×7 for one month, and C as if he had employed £12×2 for one month. If, then, we divide £100 into $6\times 3+4\times 7+12\times 2$, or 70 parts, A must have 6×3, or 18, B must have 4×7, or 28, and C 12×2, or 24 of those parts. The shares of the three are, therefore, £$\frac{6\times 3\times 100}{6\times 3+4\times 7+12\times 2}$, £$\frac{4\times 7\times 100}{6\times 3+4\times 7+12\times 2}$, and £$\frac{12\times 2\times 100}{6\times 3+4\times 7+12\times 2}$.

EXERCISES.

A, B, and C embark £100 each in an undertaking; A placing £24. 12*s*. for $2\frac{1}{2}$ years, B £100 for 1 year, and C £12 for $1\frac{1}{2}$ years. They gain £4276. 7*s*. How much must each receive of the gain?

A, B, and C rent a house together for 2 years, at £150 per annum. A remains in it the whole time, B 16 months, and C $4\frac{1}{2}$ months. How much must each pay of the rent?

THE END.

PRINTED BY THOMAS DAVISON, WHITEFRIARS.